UNITED STATES
NAVAL POSTGRADUATE SCHOOL

CURRENT DISTRIBUTION AND DRIVING POINT IMPEDANCE FOR A RHOMBIC ANTENNA

---BY---

JESSE GERALD CHANEY

PROFESSOR OF ELECTRONICS

A REPORT
TO THE NAVY DEPARTMENT
BUREAU OF SHIPS
UPON AN INVESTIGATION CONDUCTED UNDER
BUSHIPS PROJECT ORDER NO. 10731/52.

MARCH, 1954

TECHNICAL REPORT NO. 11

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REPORT NO.8

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PREFACE

This paper constitutes the eighth report upon a project which I have undertaken, under the sponsorship of the Antenna Design Section of the Bureau of Ships, for conducting mathematical studies pertaining to the distribution of current along the legs of a rhombic antenna under varying conditions. While it may seem that I have initiated this paper at too low a level, I consider it essential in justifying the engineering approximations made herein to start with the simple classical theory of transmission lines and show the relation between this theory, the generalized circuit theory, and the Hallen integral equation theory.

In connection with my formulation of the integral equation for the current along wire *one* of the rhombic antenna, I wish to thank Professor Klamm of this department and Professor Bleick of the department of Mathematics for some helpful suggestions. In fact, Professor Klamm has a somewhat similar formulation in Cartesian coordinates of a pair of simultaneous integral equations for the current along the entire rhombus. However, at the time of this writing, it seems to us that so many approximations are required in a process of iteration for even a first order solution, that the most practical approach to the problem of obtaining an engineering approximation for the current along a rhombic antenna is that of the nonuniform line theory as proposed in this paper.

J. G. C.

ABSTRACT

From Maxwell's equations, the differential equations are derived for an open wire line. The resulting classical transmission line theory is correlated with the generalized circuit theory. In fact, it is shown that the radiation impedance of a rhombic antenna may be derived from the classical theory, and the concept of radiation impedance is clarified. It is further shown that the customary usage of the difference in scalar potential for the driving point voltage is equivalent to replacing the drop across the terminal reactance, for a circuit assumed to be in a self sustained steady state oscillation, with the generator voltage. Methods for approximating the driving point impedance of a rhombic antenna are disussed. For determining the current distribution in a rhombic antenna, the Hallén integral equation technique is briefly considered and, because of its complexity, is discarded in favor of the nonuniform transmission line technnque. The current distribution is found by dividing the rhombus into four intervals, namely, $0 \leq x \leq \lambda/2$, $\lambda/2 \leq x \leq l$, $l \leq x \leq 2l-\lambda/2$, $2l-\lambda/2 \leq x \leq 2l$, and applying the nonuniform line theory. Graphs for the radiation impedance of a rhombic antenna are presented. Also, curves for the current distribution along a single wire rhombic antenna are given.

NOMENCLATURE

$\bar{B}$ magnetic flux density vector

$\bar{E}$ electric displacement vector

$\bar{H}$ magnetic field vector

$\bar{i}$ current density vector

$\bar{A}$ vector potential

ϕ scalar potential

σ conductivity

$\epsilon_0 = 10^{-9}/36\pi$ fd./m. permittivity of free space

$\mu_0 = 4\pi(10^{-7})$ h./m. permeability of free space

$\eta_0 \doteq 120\pi$ ohms intrinsic impedance of free space

$k = 2\pi/\lambda = \omega(\mu_0\epsilon_0)^{\frac{1}{2}}$ free space wave number

$\omega = 2\pi f$ angular frequency

l length of transmission line, length of one leg of rhombic antenna

a radius of wire

ρ spacing of line

$2\phi_0$ vertex angle at driving point of rhombic antenna

s arc length coordinate along a wire

x line length coordinate

I current

V voltage

R resistance per unit length of line

G conductance per unit length of line

L inductance per unit length of line

C capacitance per unit length of line

F form factor for line, contribution to generalized voltage by vector potential

f normalized current function

f_m spatial root mean square of normalized current function

f_0 normalized current in terminal impedance

Z_i internal impedance per unit length of line

Z_r radiation impedance

Z_0 characteristic impedance of line

α attenuation constant

γ propagation constant

B difference between actual scalar potential and the value given by integrating the charge density over the wires of a line

$\underset{\sim}{C}_e$ end capacitance

$\underset{\sim}{V}$ generalized voltage

$\underset{\sim}{L}$ inductance per unit length for generalized voltage

$\underset{\sim}{C}$ capacitance per unit length for generalized voltage

K_{av} average characteristic impedance of nonuniform transmission line

M, N parameters in nonuniform line theory

$j = (-1)^{\frac{1}{2}}$ time phasor operator

$Z^{\perp}$ difference between actual series impedance per unit length and nominal series impedance per unit length

$Y^{\perp}$ difference between actual shunt admittance and nominal or average shunt admittance per unit length

log common logarithm

ln natural logarithm to base e=2.71828

$\oint$ closed line integral

$\nabla = \bar{a}_x \frac{\partial}{\partial x} + \bar{a}_y \frac{\partial}{\partial y} + \bar{a}_z \frac{\partial}{\partial z}$ vector operator del

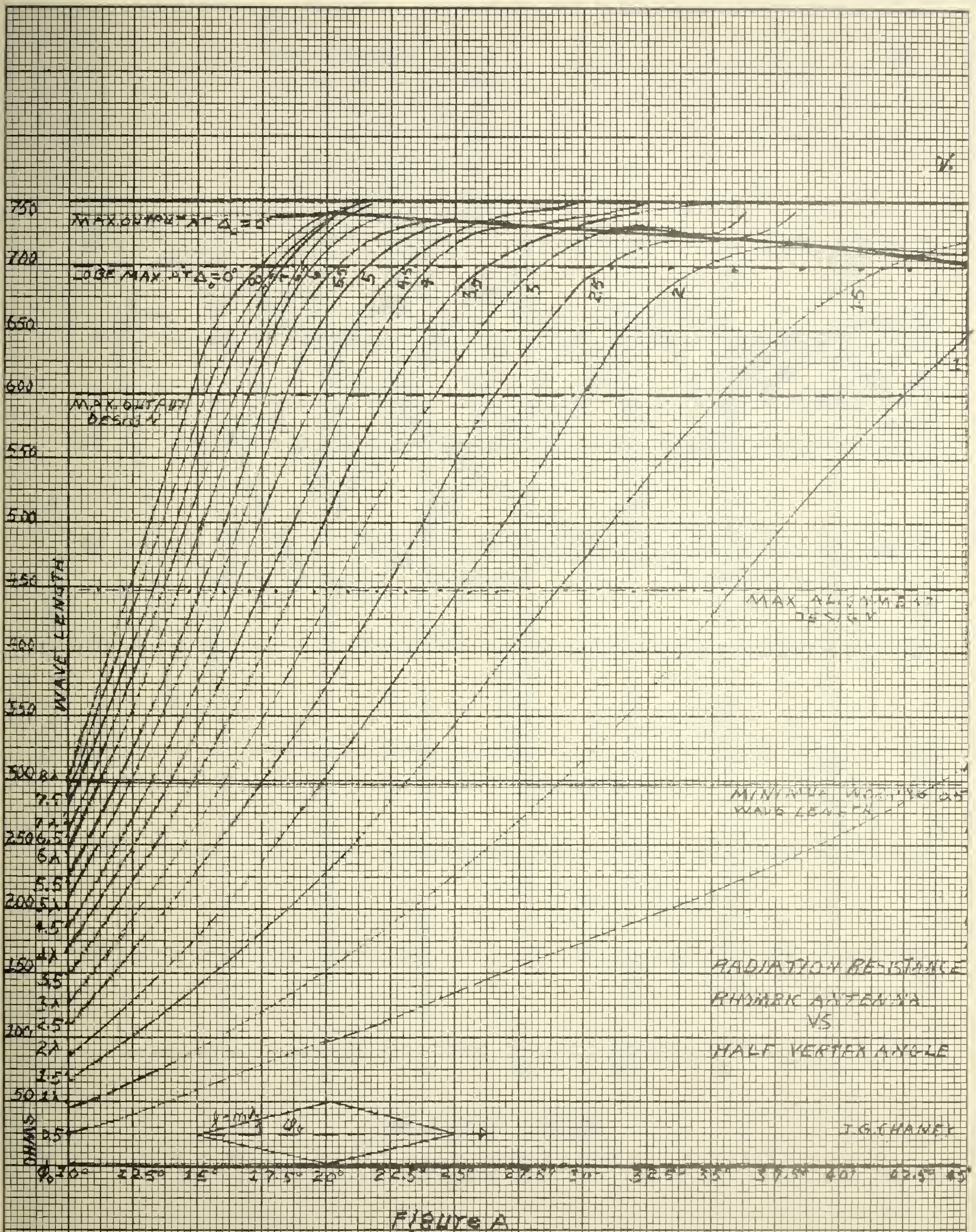
V.
MAX. OUTPUT AT Δ = 0°
LOBE MAX AT Δ = 0°
MAX. OUTPUT DESIGN
WAVE LENGTH
MAX ALIGNMENT DESIGN
OHMS
RADIATION RESISTANCE
RHOMBIC ANTENNA
VS
HALF VERTEX ANGLE
J.G. CHANEY

FIGURE A

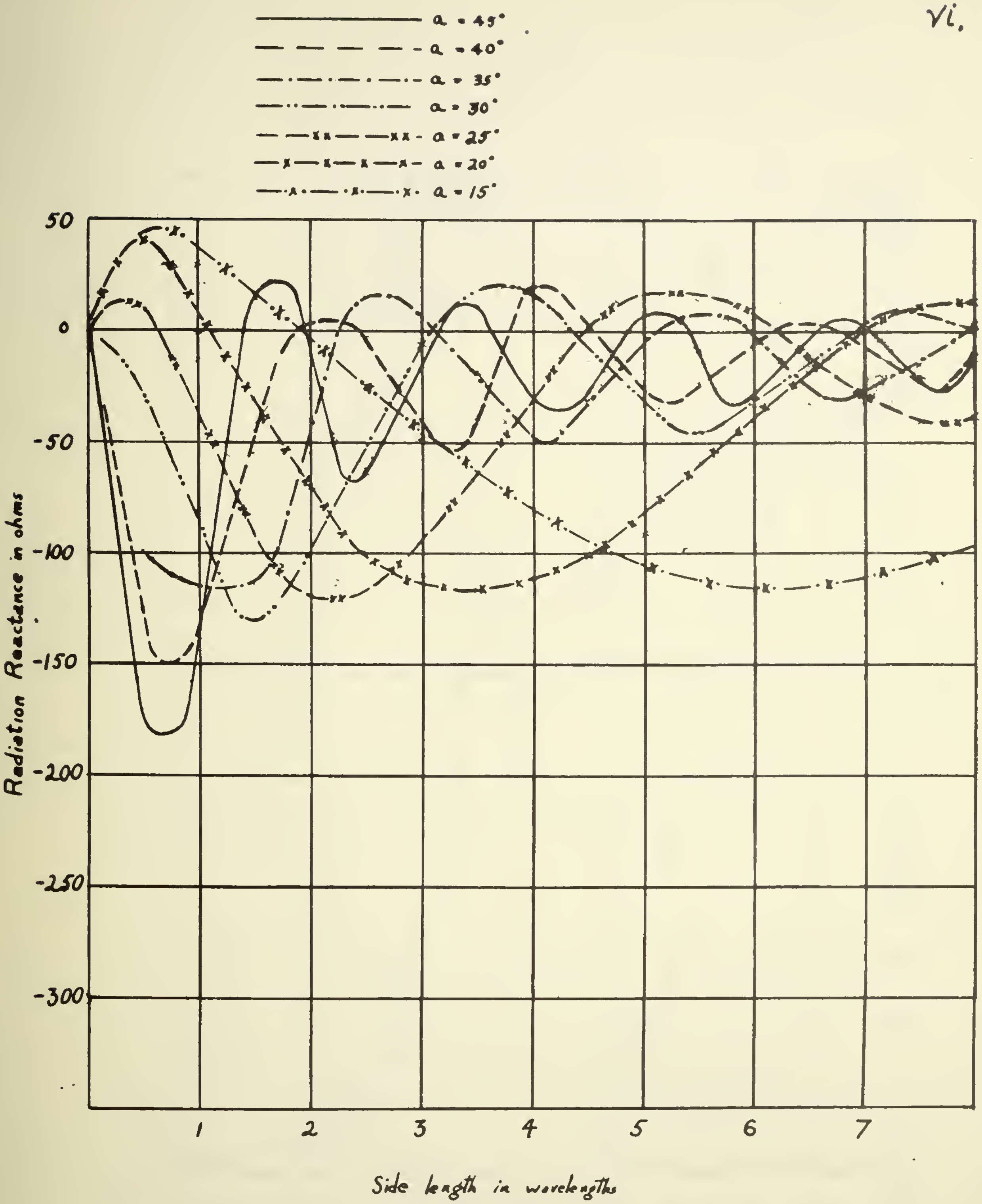
a = 45°
a = 40°
a = 35°
a = 30°
a = 25°
a = 20°
a = 15°
50
0
-50
-100
-150
-200
-250
-300
Radiation Reactance in ohms
1
2
3
4
5
6
7
Side length in wavelengths

Figure - B

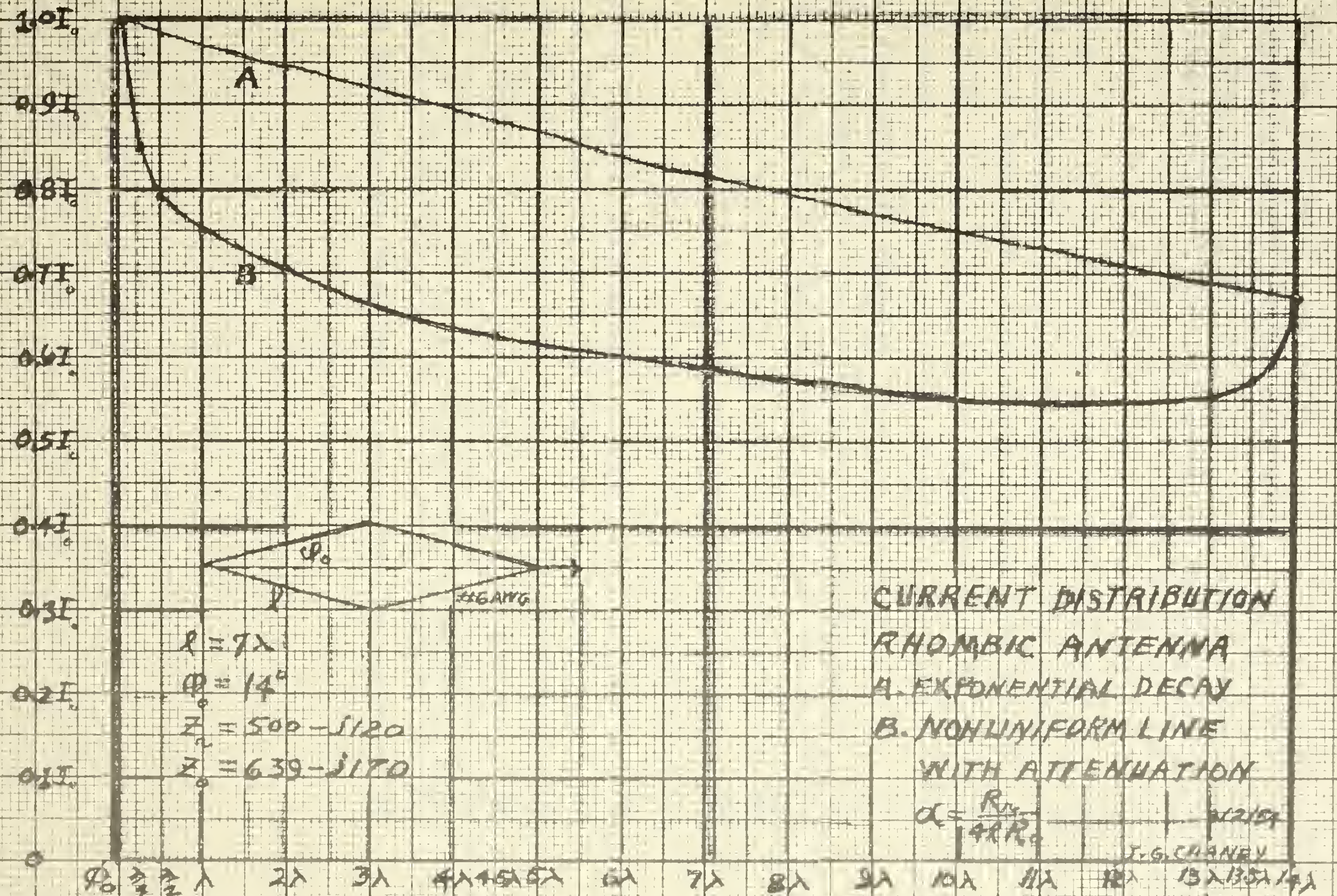
1.0I₀
0.9I₀
0.8I₀
0.7I₀
0.6I₀
0.5I₀
0.4I₀
0.3I₀
0.2I₀
0.1I₀
0
A
B
λ
2λ
3λ
6λ
7λ
8λ
9λ
10λ
11λ
12λ
14λ
l = 7λ
Φ₀ = 14°
Z_r = 500 - j120
Z₀ = 639 - j170
CURRENT DISTRIBUTION
RHOMBIC ANTENNA
A. EXPONENTIAL DECAY
B. NONUNIFORM LINE
WITH ATTENUATION

Figure C

TABLE OF CONTENTS

CURRENT DISTRIBUTION AND DRIVING POINT IMPEDANCE FOR A RHOMBIC ANTENNA

I. INTRODUCTION

Since the radiation impedance[3,6] *of a rhombic antenna* has been derived from an application of the generalized circuit theory[2], it is desirable to enter into a detail analysis of the relation between the generalized circuit theory as applied to open wire transmission lines[1] and the customary classical theory of such lines. Such an analysis tends to clarify the significance of the term *radiation reactance*, brings out the types of approximations involved, and suggests a method for relating the radiation impedance of a line to the driving point impedance of the line. Afterwards, the radiation impedance of a rhombic antenna may be used in connection with Schelkunoff's nonuniform transmission line theory[7,9] for approximately determining the current distribution and driving point impedance of the rhombic antenna.

II. DIFFERENTIAL EQUATIONS

Since the generalized circuit theory[2] is derived from the complex form of Maxwell's equations, it seems necessary to review the derivation of the differential equations for the transmission line as derived directly from Maxwell's equations, and to point out certain relationships, even though such derivations are well known[9] The derivations will be from the instantaneous form of Maxwell's equations, namely,

$$\nabla\cdot\bar{B} = 0 \qquad \nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t} \qquad \bar{D} = \epsilon\bar{E} \qquad \bar{H} = \nabla\times\bar{A}$$

$$\nabla\cdot\bar{D} = \rho \qquad \nabla\times\bar{H} = \bar{\imath} + \frac{\partial\bar{D}}{\partial t} \qquad \bar{B} = \mu\bar{H} \qquad \bar{E} = (\nabla\nabla\cdot + k^2)\bar{A}/j\omega\epsilon$$

Suppose the transmission line consists of two parallel wires each of radius a and of length l, the wires having an axial spacing ρ. Consider an incremental length Δs *of one wire* (*Fig.1*). Now take the surface inte-

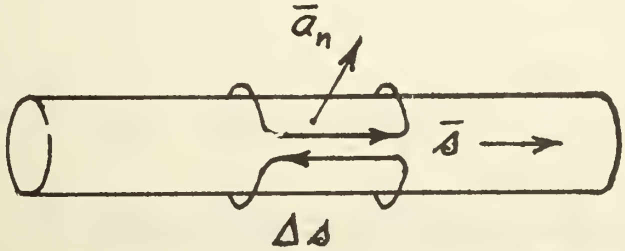

Fig. 1. Path for line integral of magnetic field vector.

gral[9] of $curl\bar{H}$ over this length, that is,

$$\int_S \nabla\times\bar{H}\cdot d\bar{S} = \oint\bar{H}\cdot d\bar{r} = \int_S (\bar{i} + \frac{\partial\bar{D}}{\partial t})\cdot d\bar{S} \qquad (1)$$

Since the longitudinal portion of the path for the line integral is traversed in each sense,

$$\oint[\bar{H}(s+\Delta s) - \bar{H}(s)]\cdot d\bar{r} = (\sigma+\epsilon\frac{\partial}{\partial t})\int_S E_n\ dS \qquad (2)$$

or

$$-I(s+\Delta s) + I(s) = \Delta s(\sigma+\epsilon\frac{\partial}{\partial t})\int_0^{2\pi}E_{mn}\,a\,d\phi \qquad (3)$$

where E_{mn} is the mean value of the normal component of the electric field vector over the interval to be evaluated at the boundary, and where σ and ϵ are for the medium surrounding the wire. The remaining integral in equation (3) is proportional to the voltage V and is a function of the radius and spacing of the wires. It may be expressed as

$$\int_0^{2\pi}E_{mn}\,a\,d\phi = F(a,\rho)^{-1}V \qquad (4)$$

where $F(a,\rho)$ is known as the form factor for the line. Hence, upon passing to the limit,

$$\frac{\partial I}{\partial s} = -F(a,\rho)^{-1}(\sigma+\epsilon\frac{\partial}{\partial t})V \qquad (5)$$

or

$$\frac{\partial I}{\partial s} = -(G + C\frac{\partial}{\partial t})V \qquad (6)$$

where G and C are the customary line parameters, that is, the conductance per unit length and the capacitance per unit length, respectively.

Perhaps equation (4) constitutes the weakest link in the derivation. This could be made stronger by retaining the electric displacement vector in equation (2), converting to the surface charge density, and then defining the capacitance per unit length. However, equation (5) serves to illustrate that the same form factor may be used for finding both G and C. Equation (6) may also be derived by integrating the equation of continuity of charge over the volume of the element.

For the second differential equation[9], $curl\,\bar{E}$ will be integrated over the surface between the two segments of wires forming the elementary length of line (*Fig. 2*). Firstly, the integration yields Faraday's law,

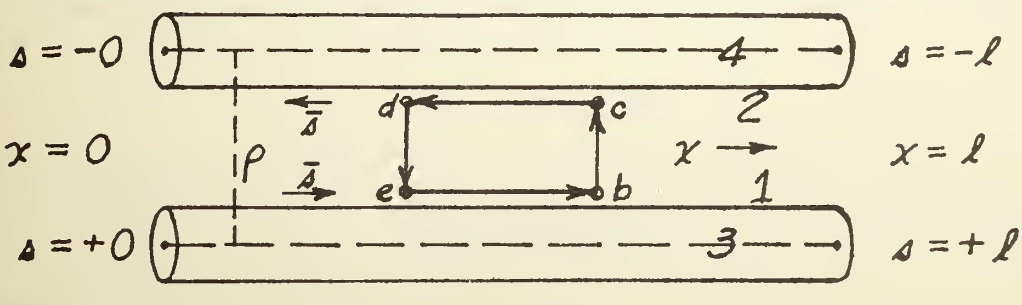

Fig. 2. Path for line integral of electric field vector.

$$\oint \bar{E}\cdot d\bar{r} = -\frac{\partial}{\partial t}(\Delta x\ L_e\ I) \qquad (7)$$

where L_e is the external inductance per unit length, defined as the magnetic flux linkage per ampere per unit length of line. Secondly,

$$\oint \bar{E}\cdot d\bar{r} = -\oint(\Delta\phi + \mu\frac{\partial \bar{A}}{\partial t}) = -\mu\frac{\partial}{\partial t}\oint \bar{A}\cdot d\bar{r} \qquad . (8)$$

In carrying out the integration for $\bar{A}$, the conventional path selections are made. Thus, path *one* and path *two* are chosen on the surfaces while paths *three* and *four* are chosen on the axes (*Fig. 2*).

The spacing ρ of the wires now will be assumed to be sufficiently small electrically for retardation to become negligible within the incremental interval. This also implies that the fields produced by the currents and charges at the more remote positions outside the interval arrive essentially in phase opposition.

In other words, the retardation in the vector magnetic potential causes a cancellation of those components of the potential produced at distances large in comparison with the spacing. If, furthermore, there exists an interval containing Δx such that its length is quite large compared to ρ and yet sufficiently small for the effects of retardation to be negligible within the interval, then the inductance per unit length may be derived from equations (7) and (8) as

$$L_e = \frac{\mu}{\pi}\ ln\ \frac{\rho}{a} \qquad (9)$$

Hence, the form factor for the open wire line is

$$F = \frac{1}{\pi}\ ln\ \frac{\rho}{a} \qquad (10)$$

It should be pointed out that if the potential $\bar{A}$ is determined by integration of the current over the entire line neglecting retardation altogether, then the form factor for positions near the ends of the line drops rapidly to one half the value given by equation (10)'.

The fact that the form factor for the capacitance per unit length is equivalent to that for the inductance per unit length may be verified after the completion of the derivations. However, such a verification is not an objective of this paper.

Returning to equation (8) and considering the integration intervalwise, upon assuming the internal impedance per unit length of line to be Z_i where operationally,

$$Z_i = R_i + L_i\frac{\partial}{\partial t} \qquad (11)$$

it is found that

$$\int_e^b \bar{E}\cdot d\bar{s} = \tfrac{1}{2}\Delta s\ Z_i I \qquad (12)$$

or since

$$-\int_e^b(\Delta\phi + \mu\frac{\partial \bar{A}}{\partial t})\cdot d\bar{s} = \phi(e) - \phi(b) - \tfrac{1}{2}\Delta s L_e\frac{\partial I}{\partial t} \qquad (13)$$

then

$$\phi(e) - \phi(b) = \tfrac{1}{2}\Delta s(Z_i + L_e\frac{\partial}{\partial t})I$$
$$= \tfrac{1}{2}\Delta s(R + L\frac{\partial}{\partial t})I \qquad (14)$$

where L is the sum of the internal and external inductance per unit length . A similar integration holds for the interval (c,d).

In other words, there is a drop in scalar potential on each wire determined by the sum of the internal impedance of the wire and one half the external inductive reactance. This shows that the external inductive reactance does actually appear in series with the internal impedance as far as the scalar potential drop along the interval is concerned.

Also, since the vector A is parallel with the wires,

$$\int_b^c \bar{E}\cdot d\bar{r} = -\int_b^c (\nabla\phi + \mu\frac{\partial \bar{A}}{\partial t})\cdot d\bar{r} = \phi(b) - \phi(c) \qquad (15)$$

and similarly for the interval (d,e). Hence, the voltage across the line is likewise a drop in scalar potential and there is no contribution to this voltage by the vector magnetic potential. Now, since

$$(\int_b^c + \int_d^e)\bar{E}\cdot d\bar{r} = [\oint - (\int_e^b + \int_c^d)]\bar{E}\cdot d\bar{r} \qquad (16)$$

it follows from equations (7), (14), (15), (16), and the corresponding equations for the intervals (c,d) and (d,e) that

$$[\phi(b) - \phi(c)] - [\phi(e) - \phi(d)] = -\Delta s\ (R + L\frac{\partial}{\partial t})\ I \qquad (17)$$

Hence

$$V(x+\Delta x) - V(x) = -\Delta x\ (R + L\frac{\partial}{\partial t})\ I \qquad (18)$$

or

$$\frac{\partial V}{\partial x} = -\ (R + L\frac{\partial}{\partial t})\ I \qquad (19)$$

Equations (6) and (19) constitute Kirchoff's circuit equations as customarily applied to the transmission line. Since the line voltages are all differences in scalar potential, and since the sum of the scalar voltage drops around the loop is given by

$$\oint\nabla\phi\cdot d\bar{r} = 0 \qquad (20)$$

even though the line integral of the electric field vector around the loop for the generalized voltage does not vanish, it follows that the conventional application of Kirchoff's voltage law is valid. Thus, it is incorrect reasoning to state, as it is sometimes done, that the external inductance is arbitrarily assumed to be in series with the internal impedance so that the line integral of the electric field vector around the loop will vanish.

III. DRIVING POINT IMPEDANCE

For comparison with the generalized circuit technique[2], the driving point impedance of a uniform line, terminated in its characteristic impedance and balanced to ground, will be found by considering the drop in complex power per unit length along the wires (*Fig. 3*).

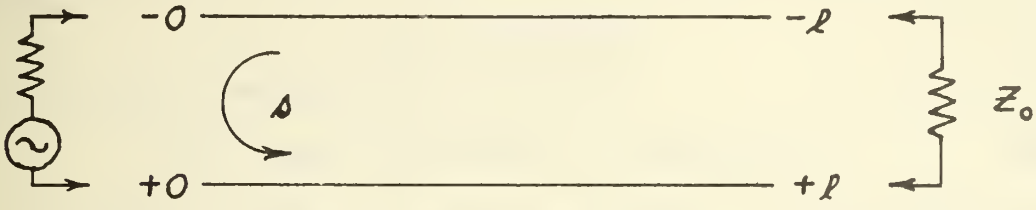

Fig. 3. Balanced line showing arc length coordinate.

Since the line is balanced and is symmetrically fed, for the arc length coordinate s satisfying $-l<s<+l$,

$$I(s) = I(-s), \qquad \phi(s) = \phi(-s)$$

and

$$V(s) = \phi(s) - \phi(-s) = 2\phi(s) \tag{21}$$

Hence, the steady state form for equations (19) and (6) may be written as[9]

$$-\frac{\partial\phi}{\partial s} = \frac{1}{2}(R + j\omega L)I \tag{22}$$

$$-\frac{\partial I}{\partial s} = \frac{1}{2}(G + j\omega C)V \tag{23}$$

Denoting the complex conjugate with an asterisk, multiplying equation (22) by the conjugate of the current, and multiplying the conjugate of equation (23) by the potential,

$$-\frac{1}{2}I^*\frac{\partial\phi}{\partial s} = \frac{1}{4}(R + j\omega L)|I|^2 \tag{24}$$

$$-\frac{1}{2}\phi\frac{\partial I^*}{\partial s} = \frac{1}{4}(G - j\omega C)|V|^2 \tag{25}$$

Integrating equation (24) by parts

$$-\frac{1}{2}\int_{-l}^{l} I(s)^*\frac{\partial\phi(s)}{\partial s}\,ds = \frac{1}{4}(R+j\omega L)\int_{-l}^{l}|I(s)|^2 ds \tag{26}$$

$$\frac{1}{4}[\phi(+0)-\phi(-0)]I(0)^* = \frac{1}{2}[\phi(l)-\phi(-l)]I(l)^* + \frac{1}{4}l(R+j\omega L)f_m^2 + \frac{1}{2}\int_{-l}^{l}\phi(s)\,\frac{\partial I(s)^*}{\partial s}\,ds \tag{27}$$

where f_m is the spatial *root mean square* of the time peak value of the

normalized spatial current distribution function.

Substituting from equation (25) and defining the characteristic impedance Z_0 of the line as the ratio of the voltage to the current in the positively travelling wave,

$$\tfrac{1}{2}V(0)I(0)^* = \tfrac{1}{2}V(l)I(l)^* + \tfrac{1}{2}l f_m^2[(R+j\omega L) + (G-j\omega C)|Z_0|^2] \qquad (28)$$

or letting

$$f_0 = |I(l)/I(0)|$$

upon multiplying equation (28) by $2/|I(0)|^2$,

$$Z_{in} = Z_0 f_0^2 + l f_m^2[(R+j\omega L) + (G-j\omega C)|Z_0|^2] \qquad (29)$$

The above procedure is exactly equivalent to adding equations (24) and (25) for the power drop per unit length of wire, and then integrating for the total power drop in the line. It is regarding the energy as circulating from the generator to the load along one wire, and back to the generator from the load along the other wire, rather than regarding it as being guided back and forth between the load and the generator by the parallel wire wave guide.

It now will be assumed that equations (6) and (19) have been solved for the steady state phasor voltage and current along the line, and that the customary values of the characteristic impedance $Z_0 = R_0 + jX_0$ and $\gamma = \alpha + j\beta$ are known[9], The attenuation constant α is customarily taken as one half the ratio of the time average power drop per unit length to the time average of the power being transmitted[9]. Perhaps it is not popularly realized that α is also given by one half the ratio in the drop per unit length of the complex power to the complex power being transmitted. That is, the ratio is the same whether or not the real parts are taken. In other words,

$$\alpha = -\frac{Re[\frac{\partial}{\partial s}(\frac{1}{2}VI^*)]}{2R_e[\frac{1}{2}VI^*)} = -\frac{\frac{\partial}{\partial s}(\frac{1}{2}VI^*)}{2\,(\frac{1}{2}VI^*)} = -\frac{\frac{\partial}{\partial s}(\frac{1}{2}\phi I^*)}{2(\frac{1}{2}\phi I^*)} \qquad (30)$$

Equations (30) may be verified either by substituting the exponential solutions and differentiating, or by taking the arithmetic mean of the propagation constant and its complex conjugate. Thus,

$$\alpha = \frac{1}{2Z_0}[(R+j\omega L) + (G-j\omega C)|Z_0|^2] = \frac{1}{2R_0}[R+G|Z_0|^2] \qquad (31)$$

Now, since the current is given by,

$$I = I_0 e^{j\omega t-\gamma|s|} \equiv I_0 f(s) e^{j\omega t} \qquad (32)$$

it follows that the mean square of the time peak normalized spatial distribution function is

$$f_m^2 = \frac{1}{2\alpha l}(1-f_0^2) \qquad (33)$$

Thus, using the ratio of the complex powers for α,

$$Z_{in} = V(0)/I(0) = Z_0 \qquad (34)$$

and using the ratio of the real powers for α,

$$\tfrac{1}{2}V(0)I(0)^*/= \tfrac{1}{2}f_0^2Z_0 + \alpha l f_m^2|I(0)|^2R_0 + \tfrac{1}{2}l f_m^2|I(0)|^2 j\omega(L-C|Z_0|^2) \qquad (35)$$

or

$$V(0)/I(0) = R_0 + j[f_0^2X_0 + l f_m^2\omega(L-C|Z_0|^2)] \qquad (36)$$

or

$$Z_{in} = R_0 + j(f_0^2X_0 + f_m^2X_r) \qquad (37)$$

where

$$X_r = \omega l(L-C|X_0|^2) \qquad (38)$$

is the radiation reactance of the line, or the reactive component of the radiation impedance. In this connection, the radiation reactance of a circuit may be defined as the ratio of that portion of the reactance produced by the external fields, as seen at the driving point, to the mean square of the time peak normalized spatial distribution function.

Of course, in the case under consideration, the spacing ρ has been selected such that retardation is negligible and the induced field has no component in time phase opposition to the current. Hence, the radiation resistance has been assumed to be negligible. Thus,

$$Z_r = j\omega X_r \qquad (39)$$

and from equation (35),

$$\tfrac{1}{2}|I(0)|^2Z_r = j\omega[\tfrac{1}{2}lL|I(0)|^2 - \tfrac{1}{2}lC|V(0)|^2] = j\omega(|U_H| - |U_E|) \qquad (40a)$$

where U_H and U_E are the energies stored in the magnetic and electric fields, respectively. Hence,

$$Z_r = j\frac{2\omega}{|I(0)|^2}(|U_H| - |U_E|) \qquad (40b)$$

or the radiation reactance is determined by the difference between the time peak energy stored in the magnetic and electric fields, assuming an unattenuated travelling wave of current to exist. This should tend to clarify the relation between the radiation reactance of a circuit and the driving point impedance.

Also, from equations (34), (37), and (40b), it follows that if there is no attenuation of the current, the radiation reactance vanishes, and the time peak energy stored in the magnetic field is equal to that stored in the electric field.

IV. RADIATING LINE

If it is assumed that the spacing of the line is not electrically negligible, the effects of retardation must be taken into consideration. To do this, the line equations will be written in terms of the retarded potentials. For, simplicity, it will be assumed that the shunt conductive

current is negligible, so that the current equation may be written in terms of the charge per unit length q simply by integrating the equation of continuity of charge over the incremental volume.

Now let path *one* lie along the surface of either wire and let path *two* lie along the axis of either wire. The differential equations then become,

$$\frac{\partial \phi_{12}(s_1)}{\partial s_1} = -\left[\tfrac{1}{2} Z_1 I(s_1) + j\omega\mu_0 \bar{A}_{12}\cdot\bar{a}_{12}\right] \tag{41}$$

$$\frac{\partial I(s_1)}{\partial s_1} = -j\omega q(s_1) \tag{42}$$

where

$$\bar{A}_{12} = \frac{1}{4\pi}\left(\int_{-l}^{-0} + \int_{+0}^{l}\right) e(r_{12})\ I(s_2)\,ds_2 \tag{43}$$

$$\phi_{12} = -1/j\omega\epsilon_0\ \nabla_1\cdot\bar{A}_{12} \tag{44}$$

$$e(r_{12}) = r_{12}^{-1}\exp(-jkr_{12}) \tag{45}$$

$$k = \omega\sqrt{\mu_0\epsilon_0} = \omega\epsilon_0\eta_0 \tag{46}$$

$$\eta_0 = 120\pi \text{ ohms} \tag{47}$$

It is a pertinent fact that the scalar potential ϕ_{12}, *as given by equations (43) and (44)*, differs from the potential ϕ_{012} *which is obtained by integrating the charge per unit length* over the wires. For

$$\phi_{012} = \frac{1}{4\pi\epsilon_0}\left(\int_{-l}^{-0} + \int_{+0}^{+l}\right) e(r_{12})\,q(s_2)\,ds_2 \tag{48}$$

and upon integrating equation (44) by parts,

$$\phi_{12} = -\frac{1}{j\omega\epsilon_0 4\pi}\left(\int_{-l}^{-0} + \int_{+0}^{+l}\right) I(s_2)\,\bar{a}_2\cdot\nabla_1 e(r_{12})\,ds_2 \tag{49}$$

$$= \frac{1}{j\omega\epsilon_0 4\pi}\left(\int_{-l}^{-0} + \int_{+0}^{+l}\right) I(s_2)\,\bar{a}_2\cdot\nabla_2 e(r_{12})\,ds_2 \tag{50}$$

$$\begin{aligned} = \frac{30}{jk}\Big\{ & I(l)\left[e(r_{1+l}) - e(r_{1-l})\right] - I(0)\left[e(r_{1+0}) - e(r_{1-0})\right] \\ & - \frac{1}{j\omega\epsilon_0 4\pi}\left(\int_{-l}^{-0} + \int_{+0}^{+l}\right) e(r_{12})\,\frac{\partial I}{\partial s_2}\,ds_2\Big\} \end{aligned} \tag{51}$$

or

$$\phi_{12} = \phi_{012} + B(s_1) \tag{52}$$

where

$$B(s_1) = \frac{30}{jk}\left\{I(l)\left[e(r_{1+l}) - e(r_{1-l})\right] - I(0)\left[e(r_{1+0}) - e(r_{1-0})\right]\right\} \tag{53}$$

with

$$B(-s_1) = -B(s_1) \tag{54}$$

Defining the line parameters L(x) and C(x) by,

$$L(x) = 2\mu_0 \bar{A}_{12} \cdot \bar{a}_1 / I(\theta) \tag{55}$$

$$1/C(x) = 2\phi_{12}/q \tag{56}$$

the voltage and current equations become

$$\frac{\partial V}{\partial x} = -(Z_1 + j\omega L)I \tag{57}$$

$$\frac{\partial I}{\partial x} = -j\omega C V \tag{58}$$

It should be pointed out that the capacitance per unit length, as defined above, includes the distributed capacitance between the ends of the wires and a given position on the line, as well as the distributed capacitance between the various elements of the line. That is, it provides for the accumulation of charges at the ends of the wires.

Thus, not only are the inductance and capacitance per unit length variable, but each is complex. The imaginary part of the inductance per unit length gives an additional drop in the voltage per unit length. Likewise, the imaginary part of the capacitance per unit length gives an equivalent conductive drop in the current per unit length. This indicates that, although the resulting radiation is not uniformly distributed, the radiation impedance is distributed and does introduce attenuation.

Returning to equations (41) and (42), the apparent drop in complex power per unit length becomes,

$$-\frac{1}{2}\frac{\partial}{\partial s_1}(\phi_{12} I^*) = \frac{1}{4} Z_i |I|^2 + j\omega\mu_0 \bar{A}_{12} \cdot \bar{I}_1^* - \frac{1}{2}\phi_{12}\frac{\partial I^*}{\partial s_1} \tag{59}$$

Equation (59) is analogous with the sum of equations (24) and (25) for the nonretarded case. Substituting from equations (43) and (51),

$$-\frac{1}{2}\frac{\partial}{\partial s_1}(\phi_{12} I^*) = \frac{1}{4} Z_i |I|^2 - \frac{1}{2}B(s_1)\frac{\partial I(s_1)^*}{\partial s_1} + \frac{15}{jk}\int_{-l}^{l} e(r_{12})\left(\frac{\partial^2}{\partial s_1 \partial s_2} - k^2 \bar{a}_1 \cdot \bar{a}_2\right) I(s_2) I(s_1)^* ds_2 \tag{60}$$

Integrating equation (60) and recalling that both $B(s_1)$ and $\frac{\partial I(s_1)^*}{\partial s_1}$ are odd functions,

$$\frac{1}{2}(\phi_{+\frac{0}{2}} - \phi_{-\frac{0}{2}}) I(0)^* - \frac{1}{2}(\phi_{+\frac{l}{2}} - \phi_{-\frac{l}{2}}) I(l)^* = \frac{1}{2} l Z_i |I(0)|^2 f_m^2 - \int_0^l B(s_1)\frac{\partial I(s_1)^*}{\partial s_1} ds_1 + \frac{15}{jk}\int_{-l}^{l}\int_{-l}^{l} e(r_{12})\left(\frac{\partial^2}{\partial s_1 \partial s_2} - k^2 \bar{a}_1 \cdot \bar{a}_2\right) I(s_2) I(s_1)^* ds_2\, ds_1 \tag{61}$$

Equation (61) provides an interesting problem in interpretation.

For $l \gg \rho$,

$$- \int_0^l B(s_1) \frac{\partial I(s_1)^*}{\partial s_1} ds_1 = 60(Cin\, k\rho + jSi\, k\rho)\,|I(0)|^2 f_m^2 \qquad (62)$$

where

$$Cin\, x + jSi\, x = \int_0^x t^{-1}[1 - exp(jt)]\, dt \qquad (63)$$

For $k\rho \ll 1$,

$$- \int_0^l B(s_1) \frac{\partial I(s_1)^*}{\partial s_1} ds_1 = [15(k\rho)^2 + j60(k\rho)]\,|I(0)|^2 f_m^2 \qquad (64)$$

In deriving equation (62), the current was approximated by assuming it to be an unattenuated travelling wave. It was then assumed that the radiated power was given approximately by using the spatial *root mean square* of the time peak value for an exponentially attenuated travelling wave. Under the same approximation, the double integral term in equation (61) vanishes. Thus, there remains,

$$\tfrac{1}{2}I(0)^*(\phi_{+02} - \phi_{-02}) = \tfrac{1}{2}I(l)^*(\phi_{+l2} - \phi_{-l2}) + \tfrac{1}{2} l f_m^2 |I(0)|^2 Z_i + |I(0)|^2 f_m^2 [15(k\rho)^2 + j\, 60 k\rho] \qquad (65)$$

The question now arising is how to define the driving point voltage. There is a temptation to define it as the difference in scalar potential. From equation (65), this would lead to a radiation impedance $\frac{1}{2}Z_r$ given by

$$\tfrac{1}{2}Z_r = 30(k\rho)^2 + j\, 120\, k\rho \qquad (66)$$

But the value of the radiation resistance given in equation (66) is one half that given by integrating the real part of the Poynting vector over a surface enclosing the wires. It is also one half that found by the generalized circuit method[1] as well as one half the value given by Storer and King[8]. Therefore, it appears that the difference in retarded scalar potential can not in general be taken as the applied voltage. This is not surprising, because of the functional relationship existing between the scalar potential and the current distribution.

Upon substituting from equation (52) into equation (65) and carrying out the integrations of the scalar potentials, it is found that the components due to the distributed charge, as given by equation (48), combine to double the radiated power term in equation (65). Also, the components produced by $B(s_1)$, as given by equation (53), give the reactive power through the end capacitances of the wires[1,2,3,4]. That is,

$$\tfrac{1}{2}I(l)^*(\phi_{+l2} - \phi_{-l2}) - \tfrac{1}{2}I(0)^*(\phi_{+02} - \phi_{-02}) \approx -j\, 30\left[\tfrac{1}{ka} - \tfrac{\cos k\rho}{k\rho} + j\left(1 - \tfrac{\sin k\rho}{k\rho}\right)\right][|I(l)|^2 - |I(0)|^2] + |I(0)|^2 f_m^2 [15(k\rho)^2 + j60\, k\rho] \qquad (67)$$

Assuming $k\rho$ is relatively small such that the the following approximations are sufficiently accurate,

$$\frac{\sin k\rho}{k\rho} \approx 1 \quad , \quad \cos k\rho \approx 1 \quad , \tag{68}$$

substitution into equation (65) with subsequent multiplication by the quantity $2/|I(0)|^2$ yields,

$$(1 + f_0^2)\frac{1}{j\omega C_e} + l f_m^2 Z_i + f_m^2[60(k\rho) + j240k\rho] = 0 \tag{69}$$

with C_e being the end capacitance[2] given by

$$1/C_e \approx \frac{2}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{\rho}\right) \tag{70}$$

The preceding analysis has taken into consideration neither the generator nor the load impedance. In reality, it has assumed the line to be in a steady state condition with the current being forced to flow around the wires through the end capacitances with no outside source of power. Equation (69) constitutes a generalization of Kirchoff's voltage law. However, it does not represent the true physical nature of the problem.

If the reactive power through the driving end capacitance is replaced by the negative of the power from the generator, that is, if it is recognized that the generator is in shunt with the end capacitance, and if the terminal impedance Z_0 is likewise considered to be in shunt with the terminal end capacitance, equation (69) becomes

$$Z_{in} = f_0^2 Z_0 + f_m^2(lZ_i + Z_r) \tag{71}$$

For determining the attenuation, the radiation resistance will be assumed to be roughly uniformly distributive. The driving point impedance will then be given by equation (37) along with the approximation for Z_0,

$$Z_0' = [(R+R_r/l) + j\omega L)(j\omega C)]^{1/2} \tag{72}$$

If exponential attenuation is assumed, and if a is taken from transmission theory as

$$a = (R+R_r)/(2lR_0) \tag{73}$$

equation (71) becomes

$$Z_{in} = R_0 + j[f_m^2(lX_i+X_r) + f_0^2 X_0] \tag{74}$$

The difference between X_0' and $f_m^2(lX_i+X_r)+f_0^2 X_0'$ may serve as a check upon the degree of the approximations involved.

V. RHOMBIC ANTENNA

The rhombic antenna (*Fig. 4*) is a nonuniform open wire transmission line. It is terminated in an impedance Z_0 which yields a current distribution that is as near to a travelling wave as it is physically possible to obtain. Each leg of the rhombus is assumed to be of length l and the vertex angle at the generator end is assumed to be $2\phi_0$. The spacings of the terminals at ± 0 and $\pm 2l$ are assumed to be such that their widths are electrically negligible.

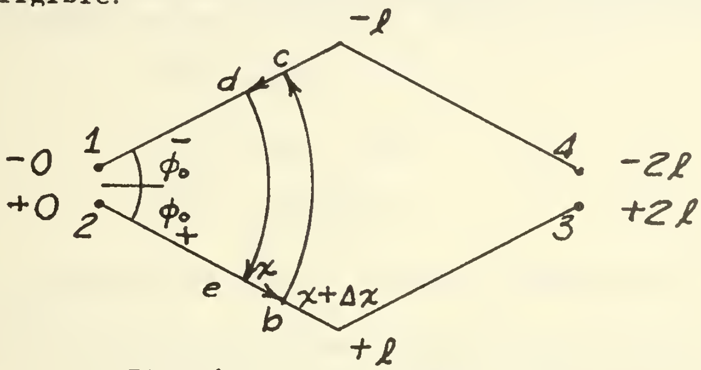

Fig. 4. Rhombic antenna

In comparing the rhombic line with the parallel line, some interesting distinctions may be observed. Perhaps the two most pertinent ones pertain to the scalar and vector potentials, respectively.

For the parallel line, it was necessary to consider the accumulation of charges at the end of the wires while finding the scalar potential. In fact, it might be said that half the radiated power was found to be due to these charges and their retardation effects. The other half was found to be due to the sum of the retardation effects due to the distributed charges as these effects appeared at each end. Thus, when the spacings of the terminals were made electrically negligible, the radiated power became negligible.

For the rhombic antenna, since the terminal spacings are electrically small, the component $B(s_1)$ vanishes in equation (52), and the scalar potential is given by equation (48) with appropriate limits of integration. That is, the scalar potential is given by only integrating the charge per unit length around the line. Also, because of the smallness of the terminal spacings, an integration of the distributed charge density for the potential drop across the terminals turns out to be the drop due to the end capacitances with no radiatuon terms remaining.

Furthermore, for the parallel line, the vector potential was found to be parallel with each wire, so that the double integral term in equation (61) vanished for the postulated current distribution. But for the rhombic line, the vector potential is parallel with neither wire.

It thus appears that the amount of power radiated by the rhombic line should be determinable from equation (61) by using just the opposite terms from those used for the parallel line. This is also logical from a physical consideration. For starting with the parallel line closely spaced, radiation occurs only as the line is distorted into the rhombus. However, a more detailed analysis should be made before a definite conclusion is reached. For the rhombic line, the voltage drop across the line is no longer only a drop in scalar potential, as it was for the parallel line, but is given by the line integral of the electric field vector. The generalized voltage $\tilde{V}$ between the wires will be defined as

$$\tilde{V} = -\int_d^e E\cdot d\bar{r} = V + F \tag{75}$$

where

$$V = \phi(e) - \phi(d) \tag{76}$$

$$F = -j\omega\mu\int_d^e \bar{A}\cdot d\bar{r} \tag{77}$$

and the path of integration is taken along the circular arc in the plane of the rhombus (*Fig. 4*).

As in the preceding analyses, arc length coordinate s will be used for integrations along individual wires, and x will be used when considering both sides of the line simultaneously. Thus, integrating the electric field vector around the closed path ($bcde$),

$$\oint\bar{E}\cdot d\bar{r} = \tilde{V}(x+\Delta x) - \tilde{V}(x) + \Delta x\, Z_i I(x) \tag{78}$$

But also,

$$\oint\bar{E}\cdot d\bar{r} = [F(x+\Delta x) - F(x)] - j\omega\mu(\Delta\bar{s}_1\cdot\bar{A}_{1m}+\Delta\bar{s}_2\cdot\bar{A}_{2m}) \tag{79}$$

where $\bar{A}_{1m}$ and $\bar{A}_{2m}$ are the mean values of the vector potential within the intervals (cd) *and* (eb) respectively. Hence, the external inductance per unit length is given by

$$\tilde{L}_e(x)I(x) = [-\frac{1}{j\omega}\frac{\partial F}{\partial x} + \mu(A_1+A_2)] \tag{80}$$

Thus, if the equation of continuity of charge,

$$\nabla\cdot\bar{\imath} = -\frac{\partial\rho}{\partial t}$$

is again integrated over the volume of an incremental cell, the rhombic line equations become,

$$\frac{\partial}{\partial x}\tilde{V}(x) = -[R + j\omega\tilde{L}(x)]I(x) \tag{81}$$

$$\frac{\partial}{\partial x}I(x) = -j\omega\tilde{C}(x)\,\tilde{V}(x) \tag{82}$$

in which the capacitance per unit length is defined by

$$\tilde{C}(x) = q(x)/\tilde{V}(x) \tag{83}$$

Multiplying equation (81) by the complex conjugate of the current, and multiplying the complex conjugate of equation (82) by the voltage, the power equation becomes

$$-\frac{1}{2}\frac{\partial}{\partial x}(\mathcal{V}I^*) = \frac{1}{2}(R+j\omega\mathcal{L})|I|^2 - j\frac{1}{2}\omega\mathcal{C}^*|\mathcal{V}|^2 \qquad (84)$$

or upon assuming the internal impedance per unit length to be constant,

$$\frac{1}{2}\mathcal{V}(0)I(0)^* = \frac{1}{2}\mathcal{V}(2l)I(2l)^* + \frac{1}{2}Z_i\int_0^{2l}|I(x)|^2dx + j\frac{\omega}{2}\int_0^{2l}[\mathcal{L}_e(x)|I(x)|^2 - C(x)^*|\mathcal{V}(x)|^2]dx \qquad (85)$$

Since the current, and hence the vector potential, is assumed to be continuous across the terminals, the terminal spacings may be taken sufficiently small for

$$\frac{1}{2}F(0)I(0)^* = 0 \qquad (86)$$

and

$$\frac{1}{2}F(l)I(l)^* = 0 \qquad (87)$$

Thus,

$$\frac{1}{2}\mathcal{V}(0)I(0)^* = \frac{1}{2}\mathcal{V}(2l)I(2l)^* + lZ_if_m^2|I_0|^2 + j\frac{1}{2}\omega\int_0^{2l}[\mathcal{L}_e(x)|I(x)|^2 - \mathcal{C}(x)^*|\mathcal{V}(x)|^2]dx \qquad (88)$$

Substituting from equations (80) and (82) into equation (88),

$$\frac{1}{2}V(0)I(0)^* = \frac{1}{2}V(2l)I(2l)^* + lZ_if_m^2|I_0|^2 - \frac{1}{2}\int_0^{2l}\frac{\partial}{\partial x}(FI^*)dx + \frac{1}{2}j\omega\mu\int_0^{2l}I(x)^*(A_1+A_2)dx - \frac{1}{2}\int_0^{2l}V(x)\frac{\partial I(x)^*}{\partial x}dx \qquad (89)$$

The first integral on the right becomes negligible because of equations (86) and (87). In fact, had the above substitution been made in equation (84), the $\frac{\partial}{\partial x}(FI^*)$ terms would have cancelled. Thus, due to the spacings of the terminals equation (84) could be derived directly by equating the tangential component of the electric field at the boundary to the internal drop in voltage, multiplying by $\frac{1}{2}I^*$, and integrating around the circuit.

Again recalling that the integrations for the potential drops $V(0)$ and $V(2l)$, determined by the distributed charge density, contribute nothing to the radiation intensity, but merely give the drop due to the end capacitances, and recalling that the physical circuit requires the generator and the terminal impedance to be in shunt with these capacitances, respectively, the driving point impedance becomes,

$$Z_{in} = Z_0f_0^2 + 2lf_m^2Z_i - \int_0^{2l}V\frac{\partial}{\partial x}I^*dx - j\omega\mu\int_0^{2l}I^*(A_1+A_2)dx \qquad (90)$$

Equation (90) can be converted into the form used in deriving the radiation impedance[3,4] of the rhombic antenna by using the arc length

coordinate s, substituting the intregal forms of the potentials, recalling that the current and vector potential are even functions while the scalar potential is an odd function, and replacing q from equations (82) and (83). Consequently,

$$Z_{in} = Z_0 f_0^2 + 2lZ_i f_m^2 + \frac{30}{jk}\int_{-2l}^{2l}\int_{-2l}^{2l} e(r_{12})\left[\frac{\partial^2}{\partial s_1 \partial s_2} - k^2 \bar{a}_1 \cdot \bar{a}_2\right] f(s_1)f^*(s_2)\, ds_1 ds_2 \qquad (91)$$

or since the axial and surface paths are interchangeable,

$$Z_{in} = Z_0 f_0^2 + 2lZ_i f_m^2 + \frac{30}{jk}\int_{-2l}^{2l}\int_{-2l}^{2l} e(r_{12})\left[\frac{\partial^2}{\partial s_1 \partial s_2} - k^2 \bar{a}_1 \cdot \bar{a}_2\right] Re[f(s_1)f^*(s_2)]\, ds_1 ds_2 \qquad (92)$$

The radiation impedance Z_r was found[3,6] by assuming to a first approximation, the current to be an unattenuated wave. Under this approximation, the parallel paths contribute nothing in equation (92). Then, since the parallel paths along a given wire are not considered, it follows that the radiation impedance should be more uniformly distributed in the case of the rhombic line than it is in the case of the parallel line. Thus, referring the radiation impedance to the spatial *root mean square* of the exponentially attenuated wave and approximately uniformly distributing the radiation impedance,

$$Z_{in} = Z_0 f_0^2 + f_m^2(2lZ_i + Z_r) \qquad (93)$$

or finding α as in equation (73) with l replaced by $2l$,

$$Z_{in} = R_0 + j[f_m^2(2lX_i + X_r) + f_0^2 X_0] \qquad (94)$$

VI. TERMINAL IMPEDANCE FOR RHOMBUS

It would be very difficult, if not impossible, to obtain a mathematically rigorous solution for that terminal impedance Z_0 which would most nearly give only a travelling wave of current on the rhombic line. Of course, because of the nature of equations (81) and (82), it is impossible to have an exact travelling wave, even if the effects of radiation are ignored.

However, since the impedance at the terminals is formally given by the ratio in the drop in scalar potential to the current at the terminals, an approximate value of Z_0 may be found by considering only the scalar voltage and applying Schelkunoff's nonuniform transmission line theory[7,8].

Accordingly, if a revised inductance per unit length $L(x)$ is defined as

$$L(x) = \mu(A_1 + A_2)/I(x) \qquad (95)$$

and a revised capacitance per unit length $C(x)$ is defined as

$$C(x) = q(x)/V(x) \tag{96}$$

upon substituting from equations (79), (80), and (95) into equation (81), and upon substituting from equations (82) and (96) into equation (82), the differential equations become,

$$\frac{\partial V(x)}{\partial x} = -[R+j\omega L(x)]I(x) \tag{97}$$

$$\frac{\partial I(x)}{\partial x} = -j\omega C(x)/V(x) \tag{98}$$

or

$$\frac{\partial V(x)}{\partial x} = -Z(x)\, I(x) \tag{99}$$

$$\frac{\partial I(x)}{\partial x} = -Y(x)\, V(x) \tag{100}$$

where $Z(x)$ and $Y(x)$ are the series impedance and shunt admittance per unit length of line, respectively. Equations (99) and (100) are in the form considered by Schelkunoff[7,8].

Since only the scalar voltage, that is, the drop in scalar potential across the wires, appears in the equations instead of the generalized voltage, the wave front may be considered to be a plane front. Thus, ignoring retardation, the nominal line parameters may be taken for the principal wave to be

$$L(x) = \frac{\mu}{\pi} \ln \frac{2x \sin\phi_0}{a} \tag{101}$$

$$C(x) = \pi\epsilon/(\ln \frac{2x \sin\phi_0}{a}) \tag{102}$$

where a is the radius of the wires.

The nominal characteristic impedance becomes,

$$K(x) = 120 \ln \frac{2x \sin\phi_0}{a} \tag{103}$$

Neglecting the internal impedance, the first order solution of equations (99) and (100) is given by Schelkunoff's theory[8] for the first Vee as

$$\begin{aligned} V(x) = {} & \frac{V_0}{K_{av}}\{[K_{av}+M(x)]\cos\beta x - N(x)\sin\beta x\} \\ & - jI_0\{[K_{av}-M(x)]\sin\beta x - N(x)\cos\beta x\} \end{aligned} \tag{104}$$

$$\begin{aligned} I(x) = {} & -j\frac{V_0}{K_{av}^2}\{N(x)\cos\beta x + [K_{av}+M(x)]\sin\beta x\} \\ & + \frac{I_0}{K_{av}}\{N(x)\sin\beta x + [K_{av}-M(x)]\cos\beta x\} \end{aligned} \tag{105}$$

where V_0 and I_0 are the driving point voltage and current, respectively. Also,

$$K_{av} = 120\left(\ln \frac{2l \sin\phi_0}{a} - 1\right) \tag{106}$$

$$M(x) = 60\left(Cin 2\beta x - 2 \ln\frac{ex}{l} \sin^2\beta x\right) \tag{107}$$

$$N(x) = 60\left(Si 2\beta x - \ln\frac{ex}{l} \sin 2\beta x\right) \tag{108}$$

$$e = 2.71828... \tag{109}$$

and for the lossless case, the propagation constant β is the same as the free space constant k.

Assuming a match to exist at the junction of the Vee's, the solution for the second Vee should be the image of that for the first Vee.

Schelkunoff and Friis[10] consider the best termination to be that which is obtained by solving for the driving point impedance of an infinite Vee. Considering the greatest variation in the voltage and current to be over the first half wave, they determine Z_0 to be [9,10]

$$Z_0 = 120 \ln \frac{\lambda \sin\phi_0}{2\pi a} - 72 - j170 \tag{110}$$

The value given in equation (110) differs from that obtained by assuming the first Vee to be terminated in its nominal characteristic impedance $K(l)$ at l and solving for the ratio V_0/Z_0. The latter solution gives the approximation

$$Z_0 = K(l)\frac{K_{av}-M}{K_{av}+M} - j2N\frac{K_{av}}{K_{av}+M} \tag{111}$$

However, equation (110) certainly is simpler to evaluate than is equation (111), and perhaps is more accurate.

VII. THE HALLEN METHOD

Since the scalar potential is given by the integration of the charge density around the rhombus, equations (82) and (98) may be integrated into equation (44), the equation of continuity of potential. Then, substituting for the scalar potential in either equation (81) or (97) and converting to arc length coordinates, the differential equations for the rhombus reduce to

$$\frac{\eta_0}{jk}\, \bar{a}_t \cdot (\nabla\nabla\cdot\bar{A} + k^2\bar{A}) = \tfrac{1}{2}\, Z_i I \tag{112}$$

Equation (112) is the basic equation for the Hallén method of iteration[5].

Theoretically, the solution of equation (112) should give a more accurate expression for the current than that given by equation (105). This follows from the fact that even though the solution of equation (112) is also a solution of equations (99) and (100), the Hallen method of iteration after converting into an integral equation, does not neglect the effects of radiation. But in the Schelkunoff method, the effects of

radiation are neglected in the line parameters given by equations (101), (102), and (103). These parameters are actually complex and are functions of the current and charge distributions. However, from a practical point of view, it hardly seems worthwhile to carry through the solution by the Hallen method because of its complexity.

For example, after neglecting the internal impedance, equation (112) along wire *one* becomes,

$$4\pi\left(\frac{d^2}{d^2s}+k^2\right)(A_{11}-A_{12}\cos 2\phi_0-A_{13}+A_{14}\cos 2\phi_0) =$$
$$s\sin^2 2\phi_0\int_0^l \sigma I(\sigma)\left[\frac{e''(r_{12})}{r_{12}^2}-\frac{e'(r_{12})}{r_{12}^3}\right]d\sigma$$
$$+\cos 2\phi_0\left[\int_0^l I(\sigma)e''(r_{12})d\sigma-\int_{-2l}^{l} I(\sigma)e''(r_{14})d\sigma\right] \tag{113}$$
$$+(s+l)\sin^2 2\phi_0\int_{-2l}^{-l}(\sigma+l)I(\sigma)\left[\frac{e''(r_{14})}{r_{14}^2}-\frac{e'(r_{14})}{r_{14}^3}\right]d\sigma$$

where the subscripts on A indicate the tangential components at wire *one* due to the various wires, the subscripts on r indicate the distances between positions on wire *one* and positions on the various wires, and the primes indicate differentiations with respect to the arguement.

Hence, even though from physical symmetry, the four equations should be reducible to two equations, one for the first Vee and one for the second Vee, the resulting pair of simultaneous integral equations are still quite unwieldy. Thus, the most accurate solution for the current distribution that seems practical is that given by Schelkunoff's theory.

VIII. CURRENT ALONG LEGS OF RHOMBUS

The integral equations corresponding to equations (99) and (100) are given by Schelkunoff as[7,9],

$$V(x) = V_0(x) - \int_0^x Z^{\perp}(\xi)I(\xi)\cosh\gamma(x-\xi)d\xi + K_{av}\int_0^x Y^{\perp}(\xi)V(\xi)\sinh(x-\xi)\gamma\, d\xi \tag{114}$$

$$I(x) = I_0(x) + \frac{1}{K_{av}}\int_0^x Z^{\perp}(\xi)I(\xi)\sinh\gamma(x-\xi)d\xi - \int_0^x Y^{\perp}(\xi)V(\xi)\cosh\gamma(x-\xi)d\xi \tag{115}$$

where $Z(x)$, $Y(x)$, $K(x)$ *and* $K_{av}(x)$ *are determined* from equations (101), (102), (103) and (106), respectively, and where

$$Z^{\perp}(x) = Z(x)-Z_{av} \tag{116}$$

$$Y^{\perp}(x) = Y(x)-Y_{av} \tag{117}$$

with

$$Z_{av} = \frac{j\omega\mu}{\pi}\left[\ln\frac{2l\sin\phi_0}{a} - 1\right] \tag{118}$$

$$Y_{av} = Z_{av}/K_{av}^2 \tag{119}$$

$$\gamma^2 = Y_{av}Z_{av} \tag{120}$$

The voltage and current $V_0(x)$ and $I_0(x)$ represent the solution of the uniform line whose line parameters are the average values indicated above. That is,

$$V_0(x) = V_0 \cosh\gamma x - K_{av} I_0 \sinh\gamma x \qquad (121)$$

$$I_0(x) = -\frac{V_0}{K_{av}} \sinh\gamma x + I_0 \cosh\gamma x \qquad (122)$$

The first order solution is given by substituting from equations (121) and (122) into equations (114) and (115). The accuracy of the resulting solution depends upon the degree of nonuniformity of the line. In equations (104), (105), and (111), the average line was taken for the entire Vee. However, more accurate results are obtained by breaking the rhombic line into intervals. Hence, equation (110) is obtained by considering only the first half wave length, and by assuming the ratio of the voltage to the current at $x = \lambda/2$ to be given by the nominal characteristic impedance $K_1 = K(\lambda/2)$ of equation (103).

Within the interval $0 \leq x \leq \lambda/2$, the first order solution for the current is,

$$I(x)/I_0 = \left[\cosh\gamma x - \frac{Z_0}{K_{1av}}\sinh\gamma x\right] - \frac{M_1(x)}{K_{1av}}\left[\cosh\gamma x + \frac{Z_0}{K_{1av}}\sinh\gamma x\right] - j\frac{N_1(x)}{K_{1av}}\left[\frac{Z_0}{K_{1av}}\cosh\gamma x + \sinh\gamma x\right] \qquad (123)$$

with

$$Z_0 = 120 \ln\frac{\lambda \sin\phi_0}{a} - 292 - j170 \qquad (124)$$

$$K_{av} = 120 \ln\frac{\lambda \sin\phi_0}{a} - 120 \qquad (125)$$

$$M_1(x)/60 = j2\beta\int_0^x \ln\frac{2e\xi}{\lambda}\sinh 2\gamma\xi \, d\xi \qquad (126)$$

$$N_1(x)/60 = -2\beta\int_0^x \ln\frac{2e\xi}{\lambda}\cosh 2\gamma\xi \, d\xi \qquad (127)$$

Substituting $\alpha + j\beta = \gamma$ into equations (126) and (127), expanding, and noting that

$$\sinh 2\alpha\xi \approx 0, \qquad \cosh 2\alpha\xi \approx 1 \qquad (128)$$

the coefficients $M_1(x)$ and $N_1(x)$ become approximately

$$\frac{M_1(x)}{60} = Cin 2\beta x - 2\ln\frac{2ex}{\lambda}\sin^2\beta x \qquad (129)$$

$$\frac{N_1(x)}{60} = Si 2\beta x - \ln\frac{2ex}{\lambda}\sin 2\beta x \qquad (130)$$

The significance of the various terms in equation (123) may be obtained by permitting K_{1av} to become very nearly equal to Z_0. Thus, in such a case,

$$I(x)/I_0 \approx e^{-\gamma x} - \frac{M_1(x)+jN_1(x)}{K_{1av}} e^{\gamma x} \qquad (131)$$

and it is seen that the first bracket represents the outward travelling wave, and that the remaining terms represent a reflected wave of varying phase which is continuously being generated by the voltage because of the nonuniformity of the line, or because of the variable characteristic impedance of the line.

Within the second interval, $\lambda/2 \leq x \leq l$, equations (121) and (122) become,

$$V_0(x) = V(\lambda/2)\cosh\gamma(x-\lambda/2) - K_{2av} I(\lambda/2)\sinh\gamma(x-\lambda/2) \qquad (132)$$

$$I_0(x) = - V(\lambda/2)/K_{2av} \sinh\gamma(x-\lambda/2) + I(\lambda/2)\cosh\gamma(x-\lambda/2) \qquad (133)$$

with

$$K_{2av} = 120(\ln\frac{2l\sin\phi_0}{a} - 1) + 120 \frac{\lambda}{2l-\lambda} \ln 2l/\lambda \qquad (134)$$

and with similar expressions for the series impedance per unit length, that is,

$$Z(x) = j\omega\mu/\pi \ln(2x\sin\phi_0/a) \qquad (135)$$

$$Z_{av}(x) = j\omega\mu/\pi(\ln 2l\sin\phi_0/a - 1) + 120 \frac{\lambda}{2l-\lambda} \ln 2l/\lambda \qquad (136)$$

Recalling that the ratio of the voltage to the current at $x=\lambda/2$ was taken to be approximately equivalent to the nominal characteristic impedance K_1 at that point, upon substituting into equations (114) and (115), expanding and carrying out the integrations, the equation for the current within the second interval becomes,

$$\begin{aligned} I(x)/I(\lambda/2) = &[\cosh\gamma(x-\lambda/2) - K_1/K_{2av} \sinh\gamma(x-\lambda/2)] \\ &- \frac{M_2(x)}{K_{2av}}[\cosh\gamma(x-\lambda/2) + \frac{K_1}{K_{2av}}\sinh\gamma(x-\lambda/2)] \\ &- j \frac{N_2(x)}{K_{2av}}[\frac{K_1}{K_{2av}}\cosh\gamma(x-\lambda/2) + \sinh\gamma(x-\lambda/2)] \end{aligned} \qquad (137)$$

with

$$M_2(x) = M_1(x) - M_1(\lambda/2) - 120 \sin^2\beta x \ (\frac{\lambda}{2l-\lambda} \ln 2l/\lambda) \qquad (138)$$

$$N_2(x) = N_1(x) - N_1(\lambda/2) - 60 \sin 2\beta x \ (\frac{\lambda}{2l-\lambda} \ln 2l/\lambda) \qquad (139)$$

Since

$$M_2(\lambda/2) = N_2(\lambda/2) = 0 \qquad (140)$$

equation (137) yields the required continuity in the current from the first interval to the second interval.

The preceding procedure may be repeated for the second Vee. However, the coefficients become somewhat more complicated.

Again dividing the Vee into two intervals with the point of division being a half wave length from the terminals, and assuming that the ratio of the voltage to the current is approximately equal to the nominal characteristic impedance at the junction of the Vee's and at $2l-\lambda/2$, and taking

$$V_0(x) = V(l)\cosh\gamma(x-l) - K_{3av}I(l)\sinh\gamma(x-l) \tag{141}$$

$$I_0(x) = -\frac{V(l)}{K_{3av}}\sinh\gamma(x-l) + I(l)\cosh\gamma(x-l) \tag{142}$$

with

$$K_{3av} = K_{2av}, \qquad K_2 = 120\ ln\ 2l\sin\phi_0/a \tag{143}$$

substituting into equations (114) and (115), one obtains for the interval $l \leq x \leq \lambda/2$,

$$\begin{aligned} I(x)/I(l) = {} & [\cosh\gamma(x-l) - K_2/K_{2av}\sinh\gamma(x-l)] \\ & - M_3(x)/K_{2av}[\cosh\gamma(x-l) + K_2/K_{2av}\sinh\gamma(x-l)] \\ & - jN_3(x)/K_{2av}[K_2/K_{2av}\cosh\gamma(x-l) + \sinh\gamma(x-l)] \end{aligned} \tag{144}$$

with

$$\begin{aligned} M_3(x) = {} & 60\{\sin 2\beta l\ [Si2\beta l - Si2\beta(2l-x)] + \cos 2\beta l[Ci2\beta l - Ci2\beta((2l-x))] \\ & + \cos 2\beta(x-l)\ ln\frac{2l-x}{l} - \frac{4l}{2l-\lambda}\ ln\frac{2l}{\lambda}\ \sin^2\beta(x-l)\} \end{aligned} \tag{145}$$

$$\begin{aligned} N_3(x) = {} & 60\{\cos 2\beta l[Si2\beta l - Si2\beta(2l-x)] - \sin 2\beta l[Ci2\beta l - Ci2\beta(2l-x)] \\ & - \sin 2\beta(x-l)\ ln\frac{2l-x}{l} - \frac{2l}{2l-\lambda}\ ln\frac{2l}{\lambda}\ \sin 2\beta(x-l)\} \end{aligned} \tag{146}$$

As before,

$$M_3(l) = N_3(l) = 0 \tag{147}$$

and the equations yield the continuity of the current at the junction.

For the fourth interval, $2l-\lambda/2 \leq x \leq 2l$, choose

$$V_0(x) = V(2l-\lambda/2)\cosh\gamma(x-2l+\lambda/2) - K_{4av}I(2l-\lambda/2)\sinh\gamma(x-2l+\lambda/2) \tag{148}$$

$$I_0(x) = -\frac{V(2l-\lambda/2)}{K_{4av}}\sinh\gamma(x-2l+\lambda/2) + I(2l-\lambda/2)\cosh\gamma(2l-\lambda/2) \tag{149}$$

with

$$K_{4av} = K_{1av}, \qquad K_3 = K_1 \tag{150}$$

and substitute into the integral equations. The current for the fourth

interval becomes,

$$I(x)/I(2l-\lambda/2) = [\cosh\gamma(x-2l+\lambda/2) - K_1/K_{1\,av}\sinh\gamma(x-2l+\lambda/2)]$$
$$-\frac{M_4(x)}{K_{1\,av}}[\cosh\gamma(x-2l+\lambda/2) + K_1/K_{1\,av}\sinh\gamma(x-2l+\lambda/2)] \qquad (151)$$
$$-j\frac{N_4(x)}{K_{1\,av}}[K_1/K_{1\,av}\cosh\gamma(x-2l+\lambda/2) + \sinh\gamma(x-2l+\lambda/2)]$$

with

$$M_4(x) = M_1(2l-x) - M_1(\lambda/2) \qquad (152)$$

$$N_4(x) = N_1(2l-x) - N_1(\lambda/2) \qquad (153)$$

and a check shows it to be continuous at $x = 2l - \lambda/2$.

IX. CONCLUSION

In conclusion, a summary of the steps recommended for approximately determining the driving point impedance and the current distribution for a rhombic antenna will be given.

1. *Assume an unattenuated travelling wave of current and compute the nominal radiation impedance by the generalized circuit method. Curves for this impedance are given at beginning of this paper.*

2. *Assume the radiation impedance is uniformly distributed around the rhombus. This is equivalent to using the spatial root mean square value of an exponentially damped travelling wave for the unattenuated wave used in deriving the radiation impedance.*

3. *Compute the terminal impedance,*

$$Z_0 = R_0 + jX_0 = 120\log_e\frac{\lambda\sin\phi_0}{a} - 292 - j170$$

4. *Compute the attenuation constant*

$$\alpha = (2Rl+R_r)/(4R_0 l)$$

5. *Compute the square of the normalized current amplitude through the terminal impedance*

$$f_0^2 = e^{-4\alpha l}$$

6. *Compute the mean square of the normalized current amplitude,*

$$f_m^2 = (1 - f_0^2)/4\alpha l$$

7. *Compute the db loss in the terminal impedance,*

$$T = 10\log_{10}(1 - f_0^2)^{-1}$$

8 *Compute the driving point impedance,*

$$Z_{in} = R_0 + j(f_m^2 X_r + f_0^2 X_0)$$

9. *For* $0 \leq x \leq \lambda/2$ *, the current may be plotted from equation (123) with the parameters given by equations (124), (125), (129), and (130).*

10. *For* $\lambda/2 \leq x \leq l$*, the current may be plotted from equations (137) with the parameters given by equations (103), (136), (138), and (139).*

11. *For* $l \leq x \leq 2l - \lambda/2$*, the current may be plotted from equation (144) with the parameters given by equations (136), (143), (145), and (146).*

12. *For* $2l - \lambda/2 \leq x \leq 2l$*, the current may be plotted from equation (151) with the parameters given by equations (103), (125), (129), (130), (152), and (153).*

A graph showing the currrnt distribution for a typical single wire rhombic antenna is given at the beginning of this paper. For a wave angle of 10^0, this antenna has a theoretical directivity of 17.25 *db* at a frequency of 14.87 *mcps*. The termination loss is 2.66 *db*, and the power gain over a half wave dipole in the same position is 12.39 *db*.

Curve *A* for the currrnt is for the hypothesis that the current may be approximated sufficiently well by an exponentially damped travelling wave, when the radiated power is being computed. Curve *B* is for steps 10 to 13 above, assuming attenuation in the nonuniform transmission line theory with the line broken into four intervals.

It is a remarkable coincidence that the current at the terminal impedance along curve *A* is $0.6757 I_0$, whereas the current at the terminal impedance along curve *B* is $0.6751 I_0 \angle -0^\circ 40'$, an almost unbelievable coincidence. However, the input power computed by using the current and nominal characteristic impedance at a half wave length from the driving point terminals, as suggested by Schelkunoff abd Friis[10] for multiple wire rhombic antennas, is 0.43 *db* above that computed at the driving point terminals.

NOTES

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APPENDIX-A-
INDUCTANCE OF CLOSE SPACED LINE

Given (Fig. 2)

$$\Delta x\, L_e \frac{\partial}{\partial t} I(x) = 2\mu \frac{\partial}{\partial t}\left(\int_a^b \bar{A}_{13}\cdot d\bar{s}_1 - \int_d^c \bar{A}_{12}\cdot d\bar{s}_1\right)$$

with $a \ll \rho \ll l$, $K\rho \ll 1$

For $\bar{A}_{13}$, $r_{13} = \sqrt{a^2+(x_3-x_1)^2}$

and for $\bar{A}_{12}$, $r_{12} = \sqrt{\rho^2+(x_2-x_1)^2}$

Let $x_3 = x_2$

For $|x_2-x_1| \gg \rho$; $r_{12} \approx |x_2-x_1| + \frac{\rho^2}{2|x_2-x_1|}$, $r_{13} \approx |x_2-x_1|$

Hence

$$\frac{e^{-jKr_{13}}}{r_{13}} - \frac{e^{-jKr_{12}}}{r_{12}} \approx \frac{1}{|x_2-x_1|} e^{-jK|x_2-x_1|}\left(e^{-\frac{jK\rho^2}{2|x_2-x_1|}} - 1\right) \approx 0$$

For $Kr_{12} \ll 1$ but not $|x_2-x_1| \gg \rho$

$$\frac{e^{-jKr_{12}}}{r_{13}} - \frac{e^{-jKr_{12}}}{r_{12}} \approx \frac{1}{r_{13}} - \frac{1}{r_{12}}$$

Suppose x_0 is a value of $x_2 \ni x_0 \gg \rho$, $|x_0 - x_1| \gg \rho$ and ρ is sufficiently small $\ni$ if $x_1 - x_0 < x_2 < x_1 + x_0$ then $Kr_{12} \ll 1$ and

$$\Delta x\, L_e = \frac{\mu}{2\pi}\int_x^{x+\Delta x}\int_{x_1-x_0}^{x_1+x_0}\left(\frac{1}{r_{12}} - \frac{1}{r_{13}}\right) dx_2\, dx_1$$

or

$$L_e = \frac{\mu}{2\pi}\int_{x-x_0}^{x+x_0}\left(\frac{1}{\sqrt{a^2+(x_2-x)^2}} - \frac{1}{\sqrt{\rho^2+(x_2-x)^2}}\right) dx_2$$

$$= \frac{\mu}{2\pi}\left[\ln \frac{x_2 - x + \sqrt{a^2+(x_2-x)^2}}{x_2 - x + \sqrt{\rho^2+(x_2-x)^2}}\right]_{x-x_0}^{x+x_0}$$

$$= \frac{\mu}{2\pi}\left[\ln \frac{x_0+\sqrt{a^2+x_0^2}}{x_0+\sqrt{\rho^2+x_0^2}} - \ln \frac{-x_0+\sqrt{a^2+x_0^2}}{-x_0+\sqrt{\rho^2+x_0^2}}\right]$$

Since the first term is negligibly small,

$$L_e \approx \frac{\mu}{2\pi} \ln \frac{x_0 - x_0[1+(\rho/l)^2]^{1/2}}{x_0 - x_0[1+(a/l)^2]^{1/2}} \approx \frac{\mu}{\pi}\ln \rho/a$$

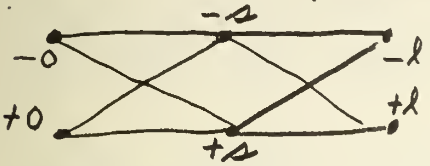
APPENDIX-B-

RADIATION IMPEDANCE OF PARALLEL LINE

Given

$$\frac{jKZ_r}{30} = \left(\int_{-l}^{-0} + \int_{+0}^{l}\right)\left[f_2(-l)e(r_{1-l}) - f_2(-\rho)e(r_{1-\rho}) + f_2(\rho)e(r_{1+\rho}) - f_2(l)e(r_{1l})\right]$$

$$\times\ \partial f_1^{*}/\partial s_1\, ds_1$$



For $s<0$

$r_{1-l} = \sqrt{(l+s)^2+a^2}$ $\quad r_{1-\rho}=\sqrt{s^2+a^2}$

$r_{1+l} = \sqrt{(l+s)^2+\rho^2}$ $\quad r_{1+\rho}=\sqrt{s^2+\rho^2}$

For $s>0$, $r_{1-l}=\sqrt{(l-s)^2+\rho^2}$, $r_{1-\rho}=\sqrt{s^2+\rho^2}$, $r_{1+\rho}=\sqrt{s^2+a^2}$

$r_{1+l}=\sqrt{(l-s)^2+a^2}$

now

$f(s)=e^{-jK|s|}$; $s<0$, $f(s)=e^{jKs}$; $s>0$, $f(s)=e^{-jKs}$

$f(s)=f(-s)$, $f'(s)=-f'(-s)$; $f'(s)^{*} = jKe^{jKs}$ for $s>0$

For $s<0$, let $s=-x$

$$\frac{Z_r}{60} = \int_0^l\left[\frac{e^{-jK(l+r_{1-l})}}{r_{1-l}} - \frac{e^{-jKr_{1-\rho}}}{r_{1-\rho}} + \frac{e^{-jKr_{1+\rho}}}{r_{1+\rho}} - \frac{e^{-jK(l+r_{1l})}}{r_{1l}}\right]e^{jKs}\,ds$$

In $e^{-jKl}\int_0^l \frac{e^{jK(s-r_{1-l})}}{r_{1-l}}\,ds$ let $z=l-s$, $s=l-z$, $r=\sqrt{z^2+\rho^2}$

$$\int_0^l \frac{e^{-jK(z+r)}}{r}\,dz = \mathrm{Ci}K(\sqrt{l^2+\rho^2}+l) - \mathrm{Ci}K\rho - j[\mathrm{Si}K(\sqrt{l^2+\rho^2}+l) - \mathrm{Si}K\rho]$$

Similarly

$$-\int_0^l \frac{e^{-jK(r_{1-\rho}-s)}}{r_{1-\rho}}\,ds = \mathrm{Ci}K(\sqrt{l^2+\rho^2}-l) - \mathrm{Ci}K\rho + j[\mathrm{Si}K\rho - \mathrm{Si}K(\sqrt{l^2+\rho^2}-l)]$$

$$\int_0^l \frac{e^{-jK(r_{1+\rho}-s)}}{r_{1+\rho}}\,ds = \mathrm{Ci}Ka - \mathrm{Ci}K(\sqrt{l^2+a^2}-l) - j[\mathrm{Si}Ka - \mathrm{Si}K(\sqrt{l^2+a^2}-l)]$$

$$-e^{-jKl}\int_0^l \frac{e^{-jK(r_{1l}-s)}}{r_{1l}}\,ds = \mathrm{Ci}Ka - \mathrm{Ci}K(\sqrt{l^2+a^2}+l) - j[\mathrm{Si}Ka - \mathrm{Si}K(\sqrt{l^2+a^2}+l)]$$

$$\frac{Z_r}{60} = \mathrm{Ci}2Kl - \mathrm{Ci}K\rho - j(\mathrm{Si}2Kl - \mathrm{Si}K\rho) - \mathrm{Ci}K\rho + \ln\frac{\gamma K\rho^2}{2l} + j\,\mathrm{Si}K\rho$$
$$+ \ln\gamma Ka - \ln\frac{\gamma Ka^2}{2l} + \ln\gamma Ka - \mathrm{Ci}2Kl + j\,\mathrm{Si}2Kl$$

where $\ln\gamma = 0.5772$ - -

$$\frac{Z_r}{120} = -\mathrm{Ci}K\rho + j\,\mathrm{Si}K\rho + \ln\gamma K\rho = \mathrm{Cin}\,K\rho + j\,\mathrm{Si}K\rho$$

$$\frac{Z_{21}}{60} = -\int_0^{l}\left[\frac{e^{jk(l-s-r_{1-l})}}{r_{1-l}} - \frac{e^{-jk(r_{1\rho}+s)}}{r_{1-\rho}} + \frac{e^{-jk(r_{1\rho}+s)}}{r_{1\rho}} - \frac{e^{jk(l-s-r_{1l})}}{r_{1l}}\right] ds$$

Let $z = l - s$, $-s = z - l$, $r_0 = \sqrt{\rho^2 + z^2}$

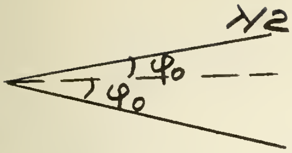
$$-\int_0^{l} \frac{e^{-jk(r_0 - z)}}{r_0}\, dz = \text{Ci}\,k\rho - \text{Ci}\,k(\sqrt{l^2+\rho^2} - l) - j[\text{Si}\,k\rho - \text{Si}\,k(\sqrt{l^2+\rho^2} - l)]$$

$$\int_0^{l} \frac{e^{-jk(r_{1-\rho}+s)}}{r_{1-\rho}}\, ds = \text{Ci}\,k\rho - \text{Ci}\,k(\sqrt{l^2+\rho^2} + l) + j[\text{Si}\,k(\sqrt{l^2+\rho^2} - \text{Si}\,k\rho]$$

$$\int_0^{l} \frac{e^{-jk(r_{1\rho}+s)}}{r_{1\rho}}\, ds = \text{Ci}\,k(\sqrt{l^2+a^2} + l) - \text{Ci}\,ka - j[\text{Si}\,k(\sqrt{l^2+a^2} - \text{Si}\,ka)]$$

$$\int_0^{l} \frac{e^{-jk(r_{1\rho}-s)}}{r_{1\rho}}\, ds = \text{Ci}\,k(\sqrt{l^2+a^2} + l) - \text{Ci}\,ka - j[\text{Si}\,k(\sqrt{l^2+a^2} - \text{Si}\,ka)]$$

APPENDIX-C-



M(x) FOR $0 < x < \lambda/2$

$$Z(x) = \frac{j\omega\mu}{\pi} \ln \frac{2x \sin\varphi_0}{a}$$

$$\frac{\lambda\pi}{j2\omega\mu} Z_{av} = \int_0^{\lambda/2} \ln \frac{2x\sin\varphi_0}{a} dx, \quad du = \frac{dx}{x}, \; v = x$$

$$= \frac{\lambda}{2}\left(\ln \frac{\lambda \sin\varphi_0}{a} - 1\right)$$

$$Z_{av} = j\omega\mu\left(\ln \frac{\lambda\sin\varphi_0}{a} - 1\right) = j\omega\mu \ln \frac{\lambda \sin\varphi_0}{a e}$$

$$Z(x) - Z_{av} = j\omega\mu \ln \frac{2ex}{\lambda}, \quad e = 2.71828\ldots$$

$$\frac{M(x)}{60} = -2\beta \int_0^x \ln \frac{2e\xi}{\lambda} \sin 2\beta\xi \, d\xi$$

$$u = \ln\frac{2e\xi}{\lambda} \qquad dv = -2\beta \sin 2\beta\xi \, d\xi$$

$$du = d\xi/\xi \qquad v = \cos 2\beta\xi - 1$$

$$\frac{M(x)}{60} = (\cos 2\beta x - 1)\ln\frac{2ex}{\lambda} + \int_0^x \frac{1 - \cos 2\beta\xi}{\xi} d\xi$$

$$= \mathrm{Cin}\, 2\beta x - 2\sin^2\beta x \ln \frac{2ex}{\lambda}$$

$$\frac{N(x)}{60} = -2\beta\int_0^x \ln\frac{2e\xi}{\lambda}\cos 2\beta\xi \, d\xi$$

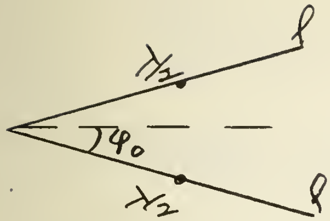
$$u = \ln\frac{2e\xi}{\lambda}, \; du = \frac{d\xi}{\xi}; \; dv = -2\beta\cos 2\beta\xi \, d\xi$$

$$v = -\sin 2\beta\xi$$

$$\frac{N(x)}{60} = -\sin 2\beta x \ln\frac{2ex}{\lambda} + \int_0^x \frac{\sin 2\beta\xi}{\xi} d\xi$$

$$= \mathrm{Si}\, 2\beta x - \sin 2\beta x \ln \frac{2ex}{\lambda}$$

APPENDIX - D -



M(X) FOR $\lambda/2 < x < l$

$$Z(x) = \frac{\omega\mu}{\pi} \ln \frac{2x \sin\varphi_0}{a}$$

$$\sin 2\beta(x - \lambda/2) = \sin 2\beta x$$

$$\frac{(l-\lambda/2)\pi}{j\omega\mu} Z_{av} = \int_{\lambda/2}^{l} \ln \frac{2x\sin\varphi_0}{a} dx, \quad du = \frac{dx}{x}, \quad V = x$$

$$= l \ln \frac{2l\sin\varphi_0}{a} - \frac{\lambda}{2} \ln \frac{\lambda \sin\varphi_0}{a} - (l - \lambda/2)$$

$$= (l-\lambda/2) \ln \frac{2l\sin\varphi_0}{ae} + \frac{\lambda}{2}\,\frac{2l}{\lambda}$$

$$\frac{M_2(x)}{6} = -2\beta \int_{\lambda/2}^{x} \ln \frac{e\xi}{l} \sin 2\beta\xi \, d\xi - \frac{2\beta\lambda}{2l-\lambda} \int_{\lambda/2}^{x} \ln\frac{2l}{\lambda} \sin 2\beta\xi \, d\xi$$

$$u = \ln \frac{e\xi}{l}, \quad du = \frac{d\xi}{\xi}, \quad dV = -2\beta \sin 2\beta\xi \, d\xi, \quad V = \cos 2\beta\xi - 1$$

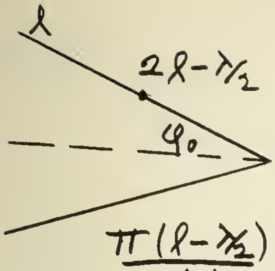
$$\frac{M_2(x)}{60} = (\cos 2\beta x - 1) \ln \frac{ex}{l} + \int_{\lambda/2}^{x} \frac{1-\cos 2\beta\xi}{\xi} d\xi + \frac{\lambda}{2l-\lambda} \ln \frac{2l}{\lambda} (\cos 2\beta x - 1)$$

$$= -2\sin^2\beta x \ln \frac{ex}{l} + \mathrm{Cin}\, 2\beta x - \mathrm{Cin}\, \beta\pi - \frac{2\lambda}{2l-\lambda} \ln \frac{2l}{\lambda} \sin^2 \beta x$$

$$\frac{N_2(x)}{60} = -2\beta \int_{\lambda/2}^{x} \ln \frac{e\xi}{l} \cos 2\beta\xi \, d\xi - \frac{2\beta\lambda}{2l-\lambda} \int_{\lambda/2}^{x} \ln \frac{2l}{\lambda} \cos 2\beta\xi \, d\xi$$

$$= -\sin 2\beta x \ln \frac{ex}{l} + \int_{\lambda/2}^{x} \frac{\sin 2\beta\xi}{\xi} d\xi - \frac{\lambda}{2l-\lambda} \ln \frac{2l}{\lambda} \sin 2\beta x$$

$$= \mathrm{Si}\, 2\beta x - \sin 2\beta x \ln \frac{ex}{l} - \mathrm{Si}\, \beta\lambda - \frac{\lambda}{2l-\lambda} \ln \frac{2l}{\lambda} \sin 2\beta x$$



APPENDIX-E-

$M(x)$ FOR $l \le x \le 2l-\lambda/2$

$$X = \frac{\omega\mu}{\pi} \ln \frac{2(2l-x)\sin\varphi_0}{a}$$

$$\frac{\pi(l-\lambda/2)}{\omega\mu} X_{av} = \int_x^{2l-\lambda/2} \ln \frac{2(2l-x)\sin\varphi_0}{a}\, dx$$

$$u = \ln \frac{2(2l-x)\sin\varphi_0}{a}, \quad du = -\frac{dx}{2l-x}; \quad dv = dx, \quad v = x-2l$$

$$\frac{\pi(l-\lambda/2)}{\omega\mu} X_{av} = -\frac{\lambda}{2}\ln\frac{\lambda\sin\varphi_0}{a} + l\ln\frac{2l\sin\varphi_0}{a} - \int_l^{2l-\lambda/2} dx$$

$$= (l-\lambda/2)\ln\frac{2l\sin\varphi_0}{a} + \frac{\lambda}{2}\ln\frac{2l}{\lambda} - (l-\lambda/2)$$

$$X_{av} = \frac{\omega\mu}{\pi}\left[\ln\frac{2l\sin\varphi_0}{ea} + \frac{\lambda}{2l-\lambda}\ln\frac{2l}{\lambda}\right]$$

$$\frac{M_3(x)}{60} = -2\beta\int_l^x \ln\frac{e(2l-\xi)}{\lambda}\sin 2\beta(\xi-l)\,d\xi - \frac{2\beta\lambda}{2l-\lambda}\ln\frac{2l}{\lambda}\int_l^x \sin 2\beta(\xi-l)\,d\xi$$

$$\eta = 2l-\xi, \quad \xi - l = l-\eta$$

$$\frac{M_3(x)}{60} = -2\beta\int_{2l-x}^{l}\ln\frac{e\eta}{l}\sin 2\beta(l-\eta)\,d\eta + \frac{\lambda}{2l-\lambda}\ln\frac{2l}{\lambda}\left[\cos 2\beta(x-l)-1\right]$$

$$u = \ln\frac{e\eta}{l}, \quad du = \frac{d\eta}{\eta}, \quad dv = -2\beta\sin 2\beta(l-\eta)\,d\eta$$
$$v = -\cos 2\beta(l-\eta)$$

$$\frac{M_3(x)}{60} = -1 + \cos 2\beta(x-l)\ln\frac{e(2l-x)}{l} + \int_{2l-x}^{l}\frac{\cos 2\beta(l-\eta)}{\eta}\,d\eta - \frac{2\lambda}{2l-\lambda}\ln\frac{2l}{\lambda}\sin^2\beta(x-l)$$

$$= -2\sin^2\beta(x-l)\left(1+\frac{\lambda}{2l-\lambda}\ln\frac{2l}{\lambda}\right) + \cos 2\beta(x-l)\ln\frac{2l-x}{l}$$
$$+ \cos 2\beta l\left[Ci\,2\beta l - Ci\,2\beta(2l-x)\right]$$
$$+ \sin 2\beta l\left[Si\,2\beta l - Si\,2\beta(2l-x)\right]$$

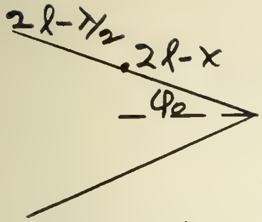
$$\frac{N_3(x)}{60} = -2\beta\int_{2l-x}^{l}\ln\frac{e\eta}{l}\sin 2\beta(l-\eta)\,d\eta - \frac{\lambda}{2l-\lambda}\ln\frac{2l}{\lambda}\sin 2\beta(x-l)$$

$$= -\sin 2\beta(x-l)\ln\frac{e(2l-x)}{l} - \int_{2l-x}^{l}\frac{\sin 2\beta(l-\eta)}{\eta}\,d\eta - \frac{\lambda}{2l-\lambda}\ln\frac{2l}{\lambda}\sin 2\beta(x-l)$$

$$= -\sin 2\beta(x-l)\left[1+\frac{\lambda}{2l-\lambda}\ln\frac{2l}{\lambda}\right] - \sin 2\beta(x-l)\ln\frac{2l-x}{l}$$

$$-\sin 2\beta l\left[Ci\,2\beta l - Ci\,2\beta(2l-x)\right] + \cos 2\beta l\left[Si\,2\beta l - Si\,2\beta(2l-x)\right]$$

$$M_3(x) = M_1(2l-x) - M_1(l) - \frac{2\lambda}{2l-\lambda}\sin^2 2\beta(2l-x)\ln\frac{2l}{\lambda}$$



APPENDIX-F-

M(x) FOR $2l-\lambda/2 \le x \le 2l$

$$\frac{\pi}{j\omega\mu} Z_1 = \ln\frac{2(2l-x)\sin\psi_0}{a} - \ln\frac{\lambda\sin\psi_0}{ae} = \ln\frac{2e(2l-x)}{\lambda}$$

$$I(x) = I_0(x-2l+\tfrac{\lambda}{2}) + \frac{j\omega\mu}{\pi Z_{ar}}\int_{2l-\lambda/2}^{x}\ln\frac{2e(2l-\xi)}{\lambda} I_0(\xi-2l+\tfrac{\lambda}{2})\sinh\gamma_0(x-\xi)\,d\xi \cdots$$

$$\frac{M_4(x)}{60} = -2\beta\int_{2l-\lambda/2}^{x}\ln\frac{2e(2l-\xi)}{\lambda}\sin 2\beta(\xi-2l+\tfrac{\lambda}{2})\,d\xi$$

$$= -2\beta\int_{2l-\lambda/2}^{x}\ln\frac{2e(2l-\xi)}{\lambda}\sin 2\beta(\xi-2l)\,d\xi$$

$$u = \ln\frac{2e(2l-\xi)}{\lambda} \qquad dV = -2\beta\sin 2\beta(\xi-2l)\,d\xi$$

$$du = -d\xi/(2l-\xi) \qquad V = \cos 2\beta(\xi-2l) - 1$$

$$\frac{M_4(x)}{60} = \ln\frac{2e(2l-x)}{\lambda}\left[\cos 2\beta(x-2l)-1\right] + \int_{2l-\lambda/2}^{x}\frac{1-\cos 2\beta(\xi-2l)}{\xi-2l}\,d\xi$$

$$= \ln\frac{2e(2l-x)}{\lambda}\left[\cos 2\beta(x-2l)-1\right] + \mathrm{Cin}\,2\beta(2l-x) - \mathrm{Cin}\,\frac{\lambda}{2}$$

$$= -2\ln\frac{2e(2l-x)}{\lambda}\sin^2\beta(2l-x) + \mathrm{Cin}\,2\beta(2l-x) - \mathrm{Cin}\,\frac{\lambda}{2}$$

$$M_4(x) = M_1(2l-x) - M_1(\lambda/2)$$

$$\frac{N_4(x)}{60} = -120\beta\int_0^{x}\ln\frac{2(\lambda/2-\eta)}{\lambda}\cos 2\beta\eta\,d\eta$$

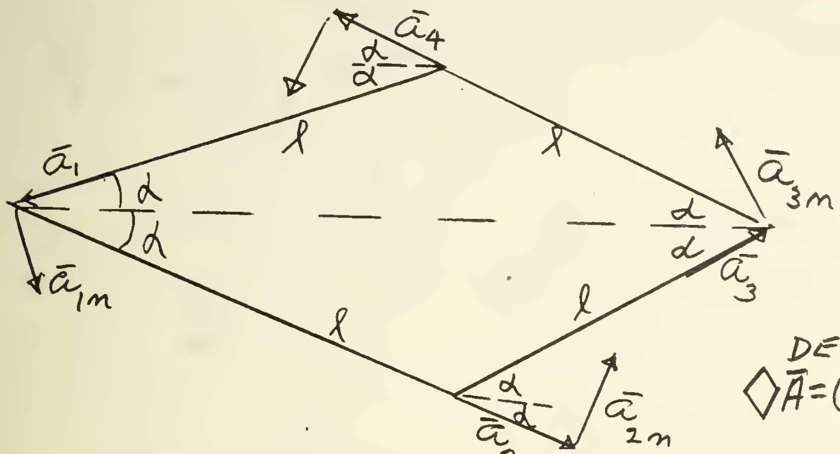
$$\frac{N_4(x)}{60} = -\ln\frac{2e(2l-x)}{\lambda}\sin 2\beta(2l-x) + \int_0^{x-2l-x}\frac{\sin 2\beta(\eta-\lambda/2)}{\eta-\lambda/2}\,d\eta$$

$$= -\ln\frac{2e(2l-x)}{\lambda}\sin 2\beta(2l-x) + \int_{\beta\lambda}^{2\beta(2l-x)}\frac{\sin t}{t}\,dt$$

$$= \mathrm{Si}\,2\beta(2l-x) - \ln\frac{2e(2l-x)}{\lambda}\sin 2\beta(2l-x) - \mathrm{Si}\,\beta\lambda$$

$$N_4(x) = N_1(2l-x) - N_1(\lambda/2)$$

APPENDIX-G
INTEGRAL EQUATION FOR WIRE ONE OF RHOMBIC ANTENNA



DEF.
$$\Diamond \bar{A} = (\nabla\nabla\cdot + K^2)\bar{A}$$

$$\nabla\cdot \bar{a}_j A_{ij} = \bar{a}_j \cdot \nabla A_{ij} \quad , \quad i,j = 1,2,3,4$$

$$\bar{a}_i \cdot \nabla\nabla\cdot \bar{a}_j A_{ij} = \frac{\partial}{\partial s_i}\left[\bar{a}_i\cdot\bar{a}_j \frac{\partial}{\partial s_i} + \bar{a}_{im}\cdot\bar{a}_j \frac{\partial}{\partial s_{im}}\right] A_{ij}$$

$$\bar{a}_j = \bar{a}_i(\bar{a}_i\cdot\bar{a}_j) + \bar{a}_{im}(\bar{a}_{im}\cdot\bar{a}_j)$$

$$(1)\begin{cases} \bar{a}_2 = -\bar{a}_1\cos 2\alpha + \bar{a}_{1m}\sin 2\alpha \,, \quad \bar{a}_3 = -\bar{a}_1 \\ \bar{a}_4 = \bar{a}_1\cos 2\alpha - \bar{a}_{1m}\sin 2\alpha \end{cases}$$

$$(2)\begin{cases} \bar{a}_1 = -\bar{a}_2\cos 2\alpha - \bar{a}_{2m}\sin 2\alpha \,, \quad \bar{a}_4 = -\bar{a}_2 \\ \bar{a}_3 = \bar{a}_2\cos 2\alpha + \bar{a}_{2m}\sin 2\alpha \end{cases}$$

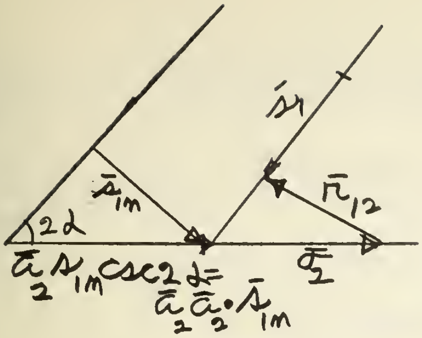
$$(3)\begin{cases} \bar{a}_2 = \bar{a}_3\cos 2\alpha - \bar{a}_{3m}\sin 2\alpha \,, \quad \bar{a}_1 = -\bar{a}_3 \\ \bar{a}_4 = -\bar{a}_3\cos 2\alpha + \bar{a}_{3m}\sin 2\alpha \end{cases}$$

$$(4)\begin{cases} \bar{a}_1 = \bar{a}_4\cos 2\alpha + \bar{a}_{2m}\sin 2\alpha \,, \quad \bar{a}_2 = -\bar{a}_4 \\ \bar{a}_3 = -\bar{a}_4\cos 2\alpha - \bar{a}_{2m}\sin 2\alpha \end{cases}$$

$$\bar{a}_i \cdot \bar{E}_{ij}\Big]_{s_i} = \frac{\eta_0}{jK}\bar{a}_i\cdot\Diamond\sum_{j=1}^{4}\bar{A}_j\Big]_{s_i} = \frac{\eta_0}{jK}\sum_{j=1}^{4}\left[\left(\frac{d^2}{ds_i^2}+K^2\right)A_{ij}\,\bar{a}_i\cdot\bar{a}_j + \frac{d^2 A_{ij}}{ds_{im}\,ds_j}\right]_{s_i} = 0$$

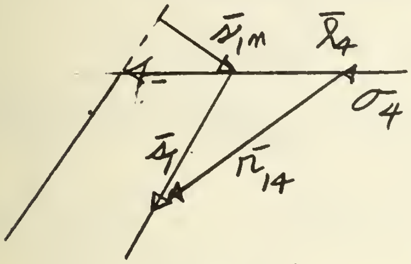
$$\frac{d^2 A_{ij}}{ds_i\,ds_j} = \bar{a}_{im}\cdot\nabla\frac{\partial A_{ij}}{\partial s_j}\Big]_{s_i}$$

$$\sum_{j=1}^{4}\left(\frac{d^2}{ds_i^2}+K^2\right)\bar{a}_i\cdot\bar{a}_j\,A_{ij} = -\bar{a}_{im}\cdot\sum_{j=1}^{4}\nabla\frac{\partial A_{ij}}{\partial s_j}\Big]_{s_i} , \quad i=1,2,3,4.$$



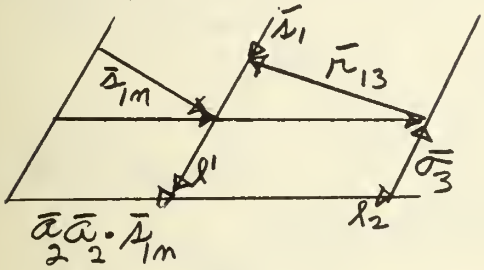
$$\bar r_{12} = (\bar l_1 - \bar s_1) + (\bar\sigma_2 - s_{1m}\csc 2\alpha\, \bar a_2)$$

$$r_{12}^2 = (l_1 - s_1)^2 - 2(l_1 - s_1)(\sigma_2 - s_{1m}\csc 2\alpha)\cos 2\alpha + (\sigma_2 - s_{1m}\csc 2\alpha)^2$$



$$\bar r_{14} = \bar l_4 - \bar\sigma_4 + \bar a_4 \bar a_4\cdot\bar s_{1m} + \bar s_1$$

$$r_{14}^2 = s_1^2 + 2s_1(l - \sigma_4 - s_{1m}\csc 2\alpha)\cos 2\alpha + (l - \sigma_4 - s_{1m}\csc 2\alpha)^2$$



$$\bar r_{13} = \bar l_1 - \bar s_1 + \bar\sigma_3 + \bar l_2 - \bar a_2 \bar a_2\cdot\bar s_{1m}$$

$$r_{13}^2 = (l_1 - \sigma_3 - s_1)^2 - 2(l_1 - \sigma_3 - s_1)(l_2 - s_{1m}\csc 2\alpha)\times \cos 2\alpha + (l_2 - s_{1m}\csc 2\alpha)^2$$

$$r_{11}^2 = (s_1 - \sigma_1)^2 + a^2$$

$$\bar A_{12} = \frac{\bar a_2}{4\pi}\int_0^l I(\sigma_2)\, e(r_{12})\, d\sigma_2 , \quad e(r_{12}) = \frac{e^{-jkr_{12}}}{r_{12}}$$

$$\frac{\partial e(r_{12})}{\partial s_1} = e'(r_{12})\frac{\partial r_{12}}{\partial s_1}$$

$$\frac{\partial^2 e(r_{12})}{\partial s_1^2} = e'(r_{12})\frac{\partial^2 r_{12}}{\partial s_1^2} + e''(r_{12})\left(\frac{\partial r_{12}}{\partial s_1}\right)^2$$

$$\frac{\partial^2 e(r_{12})}{\partial s_{1m}\partial s_1} = e'(r_{12})\frac{\partial^2 r_{12}}{\partial s_{1m}\partial s_1} + e''(r_{12})\frac{\partial r_{12}}{\partial s_1}\frac{\partial r_{12}}{\partial s_{1m}}$$

$$e'(r) = -e(r)\left(jk + \frac{1}{r}\right)$$

$$e''(r) = e(r)\left(-k^2 + j\frac{2k}{r} + \frac{2}{r^2}\right)$$

$$\frac{\partial r_{12}}{\partial s_1} = -\frac{(l_1-s_1)+(s_{1m}\csc 2\alpha-\sigma_2)\cos 2\alpha}{r_{12}}$$

$$\frac{\partial^2 r_{12}}{\partial s_1^2} = \frac{1}{r_{12}} - \frac{[(l_1-s_1)+(s_{1m}\csc 2\alpha-\sigma_2)\cos 2\alpha]^2}{r_{12}^3}$$

$$= \frac{1}{r_{12}^3}\{r_{12}^2 - [(l_1-s_1)^2 + 2(l_1-s_1)(s_{1m}\csc 2\alpha-\sigma_2)\cos 2\alpha + (s_{1m}\csc 2\alpha-\sigma_2)^2\cos^2 2\alpha]\}$$

$$= (\sigma_2 - s_{1m}\csc 2\alpha)\frac{\sin^2 2\alpha}{r_{12}^3} = \frac{1}{r_{12}^3}(s_{1m}-\sigma_2\sin 2\alpha)^2$$

$$\frac{\partial^2 e(r_{12})}{\partial s_1^2} = e''(r_{12})\left[1-\frac{(s_{1m}-\sigma_2\sin 2\alpha)^2}{r_{12}^2}\right] + \frac{e'(r_{12})}{r_{12}^3}(s_{1m}-\sigma_2\sin 2\alpha)^2$$

$$\left.\frac{\partial^2 e(r_{12})}{\partial s_1^2}\right]_{s_{1m}=0} = e''(r_{12})\left[1-\left(\frac{\sigma_2\sin 2\alpha}{r_{12}}\right)^2\right] + \frac{e'(r_{12})}{r_{12}^3}(\sigma_2\sin 2\alpha)^2$$

$$r_{12}\Big]_{s_{1m}=0} = \sqrt{(l-s_1)^2+\sigma_2^2-2\sigma_2(l-s_1)\cos 2\alpha}$$

$$\frac{\partial}{\partial s_1}\left(r_{12}\Big]_{s_{1m}=0}\right) = -\frac{(l_1-s_1)-\sigma_2\cos 2\alpha}{r_{12}\Big]_{s_{1m}=0}}$$

$$\frac{\partial^2}{\partial s_1^2}\left(r_{12}\Big]_{s_{1m}=0}\right) = \frac{1}{r_{12}}\left\{1-\frac{[(l-s_1)-\sigma_2\cos 2\alpha]^2}{r_{12}^2\Big]_{s_{1m}=0}}\right\} = \frac{1}{r_{12}^3\Big]_{s_{1m}=0}}(\sigma_2\sin 2\alpha)^2$$

$$\frac{\partial^2}{\partial s_1^2}e\left(r_{12}\Big]_{s_{1m}=0}\right) = \frac{e'\left(r_{12}\Big]_{s_{1m}=0}\right)}{r_{12}^3\Big]_{s_{1m}=0}}(\sigma_2\sin 2\alpha)^2 + e''\left(r_{12}\Big]_{s_{1m}=0}\right)\left[1-\left(\frac{\sigma_2\sin 2\alpha}{r_{12}\Big]_{s_{1m}=0}}\right)^2\right]$$

$$\therefore \left.\frac{\partial^2 e(r_{12})}{\partial s_1^2}\right]_{s_{1m}=0} = \frac{\partial^2}{\partial s_1^2}\left[e\left(r_{12}\Big]_{s_1=0}\right)\right]$$

Hence

$$\frac{\partial^2 A_{12}}{\partial s_1^2}\cos 2\alpha = \frac{\sin^2 2\alpha\cos 2\alpha}{4\pi}\int_0^l \frac{I(\sigma_2)e'(r_{12})\sigma_2^2}{r_{12}^3}d\sigma_2 - \frac{\sin^2 2\alpha\cos 2\alpha}{4\pi}\int_0^l \frac{I(\sigma_2)e''(r_{12})\sigma_2^2}{r_{12}^2}d\sigma_2 + \frac{\cos 2\alpha}{4\pi}\int_0^l I(\sigma_2)e''(r_{12})\,d\sigma_2$$

$$\frac{\partial r_{12}}{\partial s_{1m}} = \frac{1}{r_{12}}\left[(s_{1m}\csc 2\alpha - \sigma_2)\csc 2\alpha + (l_1 - s_1)\cot 2\alpha\right]$$

$$\frac{\partial^2 r_{12}}{\partial s_1 \partial s_{1m}} = -\frac{1}{r_{12}}\cot 2\alpha + \frac{1}{r_{12}^3}\Big\{\left[(l_1-s_1)+(s_{1m}\csc 2\alpha-\sigma_2)\sec 2\alpha\right]$$

$$\left[(l_1-s_1)+(s_{1m}\csc 2\alpha - \sigma_2)\cos 2\alpha\right]\Big\}\cot 2\alpha$$

$$= -\frac{1}{r_{12}^3}\cot 2\alpha\Big\{r_{12}^2 - \left[(l_1-s_1)^2 + (s_{1m}\csc 2\alpha - \sigma_2)^2\right.$$

$$\left. - (\sigma_2 - s_{1m}\csc 2\alpha)(\cos 2\alpha + \sec 2\alpha)(l - s_1)\right]\Big\}$$

$$= -\frac{1}{r_{12}^3}\cot 2\alpha(\sigma_2 - s_{1m}\csc 2\alpha)(\sec 2\alpha - \cos 2\alpha)(l - s_1)$$

$$= -(l-s_1)\frac{\sin 2\alpha}{r_{12}^3}(\sigma_2 - s_{1m}\csc 2\alpha) = \frac{l-s_1}{r_{12}^3}(s_{1m} - \sigma_2 \sin 2\alpha)$$

$$\lim_{s_{1m}\to 0}\frac{\partial^2 r_{12}}{\partial s_1 \partial s_{1m}} = \frac{\sigma_2 (s_1 - l)\sin 2\alpha}{r_{12}^3}$$

$$\lim_{s_{1m}\to 0}\frac{\partial r_{12}}{\partial s_1}\frac{\partial r_{12}}{\partial s_{1m}} = -\frac{\cot 2\alpha}{r_{12}^2}\left[(s_1 - l) + \sigma_2 \cos 2\alpha\right]\left[(s_1 - l) + \sigma_2 \sec 2\alpha\right]$$

$$= -\frac{\cot 2\alpha}{r_{12}^2}\left\{(l-s_1)^2 - (l-s_1)\sigma_2(\cos 2\alpha + \sec 2\alpha) + \sigma_2^2 - r_{12}^2 + r_{12}^2\right\}$$

$$= \frac{\cot 2\alpha}{r_{12}^2}\left\{-r_{12}^2 + (l-s_1)\sigma_2(\sec 2\alpha - \cos 2\alpha)\right\}$$

$$= -\cot 2\alpha - \frac{\sigma_2(s_1 - l)\sin 2\alpha}{r_{12}^2}$$

$$\lim_{s_{1m}\to 0}\frac{\partial^2 e(r_{12})}{\partial s_1 \partial s_{1m}} = \frac{e'(r_{12})}{r_{12}^3}\sigma_2(s_1 - l)\sin^2 2\alpha - \frac{e''(r_{12})}{r_{12}^2}\sigma_2(s_1 - l)\sin^2 2\alpha$$

$$- e''(r_{12})\cos 2\alpha$$

Therefore

$$\left(\frac{d^2}{ds_1^2} + K^2\right)A_{12}\cos 2\alpha = (s_1 - l)\sin^2 2\alpha\int_0^l \frac{I(\sigma_2)e'(r_{12})\sigma_2}{4\pi r_{12}^3}d\sigma_2$$

$$-(s_1 - l)\sin^2 2\alpha\int_0^l \frac{I(\sigma_2)e''(r_{12})\sigma_2}{4\pi r_{12}^2}d\sigma_2$$

$$-\frac{\cos 2\alpha}{4\pi}\int_0^l I(\sigma_2)e''(r_{12})d\sigma_2$$

$$r_{14}^2 = s_1^2 + 2s_1(l-\sigma_4 - s_{1m}\csc 2\alpha)\cos 2\alpha + (l-\sigma_4 - s_{1m}\csc 2\alpha)^2$$

$$\frac{\partial r_{14}}{\partial s_1} = \frac{1}{r_{14}}[s_1 + (l-\sigma_4 - s_{1m}\csc 2\alpha)\cos 2\alpha]$$

$$\frac{\partial r_{14}}{\partial s_{1m}} = 1/r_{14}[-s_1\cot 2\alpha - (l-\sigma_4 - s_{1m}\csc 2\alpha)\csc 2\alpha$$

$$= -\frac{1}{r_{14}}[s_1 + (l-\sigma_4 - s_{1m}\csc 2\alpha)\sec 2\alpha]\cot 2\alpha$$

$$\frac{\partial r_{14}}{\partial s_1}\frac{\partial r_{14}}{\partial s_{1m}} = \frac{-1}{r_{14}^2}[s_1^2 + s_1(l-\sigma_4 - s_{1m}\csc 2\alpha)(\cos 2\alpha + \sec 2\alpha)]\cot 2\alpha$$

$$= \frac{-1}{r_{14}^2}[r_{14}^2 + s_1(l-\sigma_4 - s_{1m}\csc 2\alpha)(\sec 2\alpha - \cos 2\alpha)]\cot 2\alpha$$

$$= -\cot 2\alpha - \frac{1}{r_{14}^2}[s_1(l-\sigma_4 - s_{1m}\csc 2\alpha)\sin 2\alpha]$$

$$\frac{\partial^2 r_{14}}{\partial s_{1m}\partial s_1} = -\frac{\cot 2\alpha}{r_{14}} + \frac{\cot 2\alpha}{r_{14}^3}[s_1 + (l-\sigma_4 - s_{1m}\csc 2\alpha)\cos 2\alpha][s_1 + (l-\sigma_4 - s_{1m}\csc 2\alpha)\sec 2\alpha]$$

$$= -\frac{\cot 2\alpha}{r_{14}^3}\{r_{14}^2 - [s_1^2 + s_1(l-\sigma_4 - s_{1m}\csc 2\alpha)(\cos 2\alpha + \sec 2\alpha) + (l-\sigma_4 - s_{1m}\csc 2\alpha)^2]\}$$

$$= \frac{\sin 2\alpha}{r_{14}^3}[s_1(l-\sigma_4 - s_{1m}\csc 2\alpha)]$$

$$\lim_{s_{1m}\to 0}\frac{\partial^2 e(r_{14})}{\partial s_{1m}\partial s_1}\sin 2\alpha = \sin^2 2\alpha\frac{e'(r_{14})}{r_{14}^3}s_1(l-\sigma_4) - \sin^2 2\alpha\frac{e''(r_{14})s_1(l-\sigma_4)}{r_{14}^2} - e''(r_{14})\cos 2\alpha$$

Therefore

$$\left(\frac{d^2}{ds_1^2}+K^2\right)A_{14}\cos 2\alpha = \frac{s_1\sin^2 2\alpha}{4\pi}\left[\int_0^l I(\sigma_4)\frac{e'(r_{14})}{r_{14}^3}(l-\sigma_4)\,d\sigma_4 - \int_0^l I(\sigma_4)\frac{e''(r_{14})}{r_{14}^2}(l-\sigma_4)\,d\sigma_4\right] - \frac{\cos 2\alpha}{4\pi}\int_0^l I(\sigma_4)e''(r_{14})\,d\sigma_4$$

Hence, letting $s_1 = s+l$, $s_2 = s$, $s_3 = s-l$, $s_4 = s+2l$,

$$\left(\frac{d^2}{ds^2}+K^2\right)(A_{11} - A_{12}\cos 2\alpha - A_{13} + A_{14}\cos 2\alpha) = \frac{s\sin^2 2\alpha}{4\pi}\int_0^l \sigma I(\sigma)\left[\frac{e''(r_{12})}{r_{12}^2} - \frac{e'(r_{12})}{r_{12}^3}\right]d\sigma$$
$$+ \frac{\cos 2\alpha}{4\pi}\left[\int_0^l I(\sigma)e''(r_{12})\,d\sigma - \int_{-2l}^{-l} I(\sigma)e''(r_{14})\,d\sigma\right]$$
$$+ \frac{(s+l)\sin^2 2\alpha}{4\pi}\int_{-2l}^{-l}(\sigma+l)I(\sigma)\left[\frac{e''(r_{14})}{r_{14}^2} - \frac{e'(r_{14})}{r_{14}^3}\right]d\sigma$$